BOREL-AMENABLE REDUCIBILITIES FOR SETS OF REALS

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Abstract. We show that if $\mathcal{F}$ is any "well-behaved" subset of the Borel functions and we assume the Axiom of Determinacy then the hierarchy of degrees on $\mathscr{P}({}^{\omega}\omega)$ induced by $\mathcal{F}$ turns out to look like the Wadge hierarchy (which is the special case where $\mathcal{F}$ is the set of continuous functions).

1. Introduction

Intuitively, a set A is *simpler* than — or as complex as — a set B if the problem of verifying membership in A can be reduced to the problem of verifying membership in B. In particular, if X is a Polish space and $A, B \subseteq X$, we say that A is *(continuously) reducible* to B just in case there is a continuous function f such that $x \in A$ if and only if $f(x) \in B$ for every $x \in X$, in symbols $A \leq_{\mathsf{W}} B$. (The symbol "W" is for W. Wadge who started a systematic study of this relation in his [14].) Thus in this setup continuous functions are used as *reductions* between subsets of X. The equivalence classes of the equivalence relation induced by $\leq_{\mathsf{W}}$ on the Baire space ${}^{\omega}\omega$ are called *Wadge degrees*, and the preorder $\leq_{\mathsf{W}}$ induces a partial order $\leq$ on them. Using game theoretic techniques, Wadge proved a simple but fundamental Lemma which has played a key role in various parts of Descriptive Set Theory: AD, the *Axiom of Determinacy*, implies that if $A, B \subseteq {}^{\omega}\omega$ then

$$(\star) \qquad A \leq_{\mathsf{W}} B \quad \vee \quad {}^{\omega}\omega \setminus B \leq_{\mathsf{W}} A.$$

Wadge's Lemma says that $\leq_{\mathsf{W}}$ is a semi-linear order, therefore $(\star)$ is usually denoted by $\mathsf{SLO}^{\mathsf{W}}$. Starting from this result, Wadge (and many other set theorists after him) extensively studied the preorder $\leq_{\mathsf{W}}$ and gave under $\mathsf{AD} + \mathsf{DC}(\mathbb{R})$ a complete description of the structure of the Wadge degrees (and also of the *Lipschitz degrees* — see Section 2). In [4] and [3] A. Andretta and D. A. Martin considered Borel reductions and $\boldsymbol{\Delta}^0_2$-reductions instead of continuous reductions: using topological arguments (mixed with game-theoretic techniques in the second case), they showed that the degree-structures induced by these reducibility notions look exactly like the Wadge hierarchy. Thus a natural question arises:

Question 1. Given any reasonable set of functions $\mathcal{F}$ from the Baire space into itself, which kind of structure of degrees is induced if the functions from $\mathcal{F}$ are used as reductions between sets (i.e. if we consider the preorder $A \leq_{\mathcal{F}} B \iff A = f^{-1}(B)$ for some $f \in \mathcal{F}$)?

The term "reasonable" is a bit vague, but should be at least such that all "natural" sets of functions, such as continuous functions, Borel functions and so on, are reasonable (these sets of functions will be called here *Borel-amenable* — see Section 4 for the definition).

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In this paper we will answer to the previous Question for the Borel context, i.e. when $\mathcal{F} \subseteq \mathsf{Bor}$: we will prove that under (a weakening of) $\mathsf{AD}+\mathsf{DC}(\mathbb{R})$ each of these sets of functions yields a semi-linear ordered stratification of degrees, and provides in this way a corresponding notion of complexity on $\mathscr{P}({}^{\omega}\omega)$. In particular, we will prove in Sections 4 and 5 that all these degree-structures turn out to look like the Wadge one (which can be determined as a particular instance of our results). The new key idea used in this paper is the notion of *characteristic set* $\Delta_{\mathcal{F}}$ of the collection $\mathcal{F}$, which basically contains all the subsets of ${}^{\omega}\omega$ which are simple from the "point of view" of $\mathcal{F}$ (see Section 3 for the precise definition). This tool is in some sense crucial for the study of the $\mathcal{F}$-hierarchy: in fact, if one knows the characteristic set of a Borel-amenable set of functions (even without any other information about the set $\mathcal{F}$), then one can completely describe the hierarchy of degrees induced by $\mathcal{F}$. As an application it turns out that for distinct $\mathcal{F}$ and $\mathcal{G}$, their degree-hierarchies coincide just in case $\Delta_{\mathcal{F}} = \Delta_{\mathcal{G}}$ (see Section 4 again). Moreover in Section 6 we will analyse the collection of all the Borel-amenable sets of functions and provide several examples (towards this goal we will also answer negatively to a question about generalizations of the Jayne-Rogers Theorem posed by Andretta in his [1]). Finally, in Section 7 we will show how to define some operations which allow to construct, given a certain degree in the $\mathcal{F}$-hierarchy, its successor degree(s): this will give a more combinatorial description of the degree-structure induced by $\mathcal{F}$. In a future paper we will show, building on the results obtained in this paper, how to extend our analysis of Borel reductions to a wider class of sets of functions.

The present work is, in a sense, the natural extension of [4], and we assume the reader is familiar with the arguments contained therein.

2. Preliminaries

For the sake of precision our base theory will be always $\mathsf{ZF} + \mathsf{AC}_{\omega}(\mathbb{R})$, and we will specify which auxiliary axioms are used for each statement. Nevertheless one should keep in mind that *all the results of this paper are true under* $\mathsf{AD} + \mathsf{DC}(\mathbb{R})$, or even just under $\mathsf{SLO}^{\mathsf{W}} + \neg\mathsf{FS} + \mathsf{DC}(\mathbb{R})$, where $\neg\mathsf{FS}$ is the statement[1] "there are no flip-sets" (recall that a subset F of the Cantor space ${}^{\omega}2$ is a *flip-set* just in case for every $z, w \in {}^{\omega}2$ such that $\exists! n(z(n) \neq w(n))$ one has $z \in F \iff w \notin F$). In the latter case, recall also from [2] and [3] that $\mathsf{SLO}^{\mathsf{W}} + \neg\mathsf{FS} + \mathsf{DC}(\mathbb{R})$ is equivalent to $\mathsf{SLO}^{\mathsf{L}} + \neg\mathsf{FS} + \mathsf{DC}(\mathbb{R})$. Finally, one should also observe that all the "determinacy axioms" are used in a *local way* throughout the paper: this means that our results hold for the sets in some (suitable) pointclass $\Gamma \subseteq \mathscr{P}({}^{\omega}\omega)$ whenever we assume that the corresponding axioms hold for sets in Γ. In particular, if we are content to shrink our hierarchies to the Borel subsets of ${}^{\omega}\omega$, then we only need Borel-determinacy (i.e. the determinateness of those games whose pay-off set is Borel). Therefore our results hold also in ZFC if we completely restrict our attention to the Borel context, that is if we compare only Borel sets with Borel functions.

2.1. **Notation.** Our notation is quite standard — for all undefined symbols and notions we refer the reader to the standard monograph [7] and to the survey paper [1]. The set of the natural numbers will be denoted by ω. Given a pair of set A, B, we will denote by ${}^{B}A$ the set of all the functions from B to A. In particular, ${}^{\omega}A$ will denote the collection of all the ω-sequences of elements of A, while the collection of the *finite* sequences of natural numbers will be denoted by ${}^{<\omega}\omega$ (we will refer to the length of a sequence s with the symbol $\mathrm{lh}(s)$). The space ${}^{\omega}\omega$ (the collection

[1] The axiom $\neg\mathsf{FS}$ is a consequence of both the statements "every set of reals has the Baire property" and "every set of reals is Lebesgue measurable" (hence it is also a consequence of AD), but it is weaker than them. Moreover it is consistent both with the Perfect Subset Property and its negation.

of the ω-sequences of natural numbers) is called *Baire space*, and as customary in this subject we will always identify $\mathbb{R}$ with it, that is we put $\mathbb{R} = {}^{\omega}\omega$. The Baire space is endowed with the topology induced by the metric defined by $d(x, y) = 0$ if $x = y$ and $d(x, y) = 2^{-n}$, where n is least such that $x(n) \neq y(n)$, otherwise. In particular, the basic open neighborhood of $\mathbb{R}$ are of the form $\mathbf{N}_s = \{x \in \mathbb{R} \mid s \subseteq x\}$ (for some $s \in {}^{<\omega}\omega$). Given $A \subseteq \mathbb{R}$ we put $\neg A = \mathbb{R} \setminus A$, and if $s \in {}^{<\omega}\omega$ we put $s^\frown A = \{s^\frown x \mid x \in A\}$ (when $s = \langle n \rangle$ we will simply write $n^\frown A$). Given $A_n, A, B \subseteq \mathbb{R}$ we define $\bigoplus_n A_n = \bigcup_n (n^\frown A_n)$ and $A \oplus B = \bigoplus_n C_n$, where $C_{2k} = A$ and $C_{2k+1} = B$ for every $k \in \omega$. Moreover for any $n, k \in \omega$, we put $\vec{n} = \langle n, n, n, \dots \rangle$ and $n^{(k)} = \langle \underbrace{n, \dots, n}_{k} \rangle$.

A *pointclass* (for $\mathbb{R}$) is simply a non-empty $\Gamma \subseteq \mathscr{P}(\mathbb{R})$, while a *boldface pointclass* $\boldsymbol{\Gamma}$ is a pointclass closed under continuous preimage. If $\boldsymbol{\Gamma}$ is a boldface pointclass then so is its *dual* $\breve{\boldsymbol{\Gamma}} = \{\neg A \mid A \in \boldsymbol{\Gamma}\}$. A boldface pointclass is *selfdual* if it coincides with its dual, otherwise it is nonselfdual. Finally, recall that a boldface pointclass $\boldsymbol{\Gamma}$ is said to have (or admits) a universal set if there is some $U \subseteq \mathbb{R} \times \mathbb{R}$ which is universal for $\boldsymbol{\Gamma}$ and such that the image of U under the standard homeomorphism $\mathbb{R} \times \mathbb{R} \simeq \mathbb{R}$ is in $\boldsymbol{\Gamma}$.

Let Γ be any pointclass and let $D \subseteq \mathbb{R}$. A Γ-*partition* of D is a family $\langle D_n \mid n \in \omega \rangle$ of pairwise disjoint sets of Γ such that $D = \bigcup_{n \in \omega} D_n$, and it is said to be *proper* if at least two of the D_n's are nonempty.

Let $\mathcal{F} \subseteq {}^{\mathbb{R}}\mathbb{R}$ be any set of functions. Let $f \colon \mathbb{R} \to \mathbb{R}$ be an arbitrary function and $\mathcal{C} = \langle C_n \mid n \in \omega \rangle$ be some partition of $\mathbb{R}$. We say that f is *(locally) in $\mathcal{F}$ on the partition* $\mathcal{C}$ if there is a family of functions $\{f_n \mid n \in \omega\} \subseteq \mathcal{F}$ such that $f \restriction C_n = f_n \restriction C_n$ for every n. Moreover, if $\Gamma \subseteq \mathscr{P}(\mathbb{R})$ is any pointclass, we will say that f is *(locally) in $\mathcal{F}$ on a* Γ-*partition* if there is some Γ-partition such that f is locally in $\mathcal{F}$ on it.

Given a positive real number C, we denote by $\mathsf{Lip}(C)$ the collection of all functions $f \colon \mathbb{R} \to \mathbb{R}$ which are Lipschitz with constant less or equal than C (observe that we can always assume $C = 2^k$ for some $k \in \mathbb{Z}$), and put $\mathsf{Lip} = \bigcup_{k \in \mathbb{Z}} \mathsf{Lip}(2^k)$. Moreover, since it plays a special role in the theory of reductions, we denote the set $\mathsf{Lip}(1)$ with the special symbol L. Finally, we denote by W the set of the continuous functions, by D_ξ (for some $1 \leq \xi < \omega_1$) the set of all the $\boldsymbol{\Delta}^0_\xi$-*functions* (i.e. of those f such that $f^{-1}(D) \in \boldsymbol{\Delta}^0_\xi$ for every $D \in \boldsymbol{\Delta}^0_\xi$ or, equivalently, such that $f^{-1}(S) \in \boldsymbol{\Sigma}^0_\xi$ for every $S \in \boldsymbol{\Sigma}^0_\xi$), and by Bor the set of all the Borel functions (but for simplicity of notation we will sometimes put $\mathsf{D}_{\omega_1} = \mathsf{Bor}$).

2.2. **Reducibilities.** We recall here the terminology about reducibilities for sets of reals as presented in [4] (with some minor modifications). Given a family of functions $\mathcal{F} \subseteq {}^{\mathbb{R}}\mathbb{R}$, we would like to use the functions from $\mathcal{F}$ as *reductions* and say that for every pair of sets $A, B \subseteq \mathbb{R}$, the set A is $\mathcal{F}$-*reducible to* B ($A \leq_{\mathcal{F}} B$, in symbols) if and only if there is some function $f \in \mathcal{F}$ such that $A = f^{-1}(B)$, i.e. such that $\forall x \in \mathbb{R} (x \in A \iff f(x) \in B)$. Notice that $A \leq_{\mathcal{F}} B \iff \neg A \leq_{\mathcal{F}} \neg B$. Clearly we can also introduce the strict relation corresponding to $\leq_{\mathcal{F}}$ by letting $A <_{\mathcal{F}} B \iff A \leq_{\mathcal{F}} B \wedge B \nleq_{\mathcal{F}} A$. Since in order to have degrees we would like to have $\leq_{\mathcal{F}}$ be a preorder (i.e. reflexive and transitive), *we will always assume without explicitly mentioning it that each set of functions considered is closed under composition and contains the identity function* id. Under this assumption, we can consider the equivalence relation $\equiv_{\mathcal{F}}$ canonically induced by $\leq_{\mathsf{W}}$ and call $\mathcal{F}$-*degree* any equivalence class of $\equiv_{\mathcal{F}}$ ($[A]_{\mathcal{F}}$ will denote the $\mathcal{F}$-degree of A). A set A is $\mathcal{F}$-*selfdual* if and only if $A \leq_{\mathcal{F}} \neg A$ (if and only if $A \equiv_{\mathcal{F}} \neg A$), otherwise it is $\mathcal{F}$-*nonselfdual*. Since selfduality is invariant under $\equiv_{\mathcal{F}}$, the definition can be applied

to $\mathcal{F}$-degrees as well. The *dual* of $[A]_\mathcal{F}$ is $[\neg A]_\mathcal{F}$, and a pair of distinct degrees of the form $\{[A]_\mathcal{F}, [\neg A]_\mathcal{F}\}$ is a *nonselfdual pair*. The preorder $\leq_\mathcal{F}$ canonically induces a partial order $\leq$ on the $\mathcal{F}$-degrees (the strict part of $\leq$ will be denoted by $<$). Notice also that if $\mathcal{F} \subseteq \mathcal{G} \subseteq {}^\mathbb{R}\mathbb{R}$, then the preorder $\leq_\mathcal{G}$ is coarser than $\leq_\mathcal{F}$: hence $A \leq_\mathcal{F} B \Rightarrow A \leq_\mathcal{G} B$, if A is $\mathcal{F}$-selfdual then it is also $\mathcal{G}$-selfdual, and $[A]_\mathcal{F} \subseteq [A]_\mathcal{G}$.

If $\mathcal{F}$ contains all the constant functions, then $[\mathbb{R}]_\mathcal{F} = \{\mathbb{R}\}$ and $[\emptyset]_\mathcal{F} = \{\emptyset\}$ are the $<$-least $\mathcal{F}$-degrees and form a nonselfdual pair. We say[2] that $[A]_\mathcal{F}$ is a *successor degree* if there is a degree $[B]_\mathcal{F} < [A]_\mathcal{F}$ for which there is no $C \subseteq \mathbb{R}$ such that $[B]_\mathcal{F} < [C]_\mathcal{F} < [A]_\mathcal{F}$ (such an $[A]_\mathcal{F}$ will be called *successor* of $[B]_\mathcal{F}$). If an $\mathcal{F}$-degree is not a successor and it is neither $[\mathbb{R}]_\mathcal{F}$ nor $[\emptyset]_\mathcal{F}$ (where $\mathcal{F}$ contains all the constant functions again), then we say that it is a *limit degree*. A degree $[A]_\mathcal{F}$ is of *countable cofinality* if it is minimal in the collection of the upper bounds of a family of degrees $\mathcal{A} = \{[A_n]_\mathcal{F} \mid n \in \omega\}$ each of which is strictly smaller than $[A]_\mathcal{F}$, i.e. if $[A_n]_\mathcal{F} < [A]_\mathcal{F}$ for every $n \in \omega$, and for every $[B]_\mathcal{F}$ such that $[A_n]_\mathcal{F} \leq [B]_\mathcal{F}$ (for every $n \in \omega$) we have that $[B]_\mathcal{F} \not< [A]_\mathcal{F}$ (observe that if $[A]_\mathcal{F}$ is limit the definition given here is equivalent to requiring that $[A]_\mathcal{F}$ is minimal among the upper bounds of a chain $[A_0]_\mathcal{F} < [A_1]_\mathcal{F} < \dots$). If this is not the case then $[A]_\mathcal{F}$ (is limit and) is said to be of *uncountable cofinality*.

The *Semi-Linear Ordering Principle* for $\mathcal{F}$ is

$$(\mathsf{SLO}^\mathcal{F}) \qquad \forall A, B \subseteq \mathbb{R}(A \leq_\mathcal{F} B \vee \neg B \leq_\mathcal{F} A).$$

This principle implies that if A is $\mathcal{F}$-selfdual then $[A]_\mathcal{F}$ is comparable with all the other $\mathcal{F}$-degrees, while if A and B are $\leq_\mathcal{F}$-incomparable then $\{[A]_\mathcal{F}, [B]_\mathcal{F}\}$ is a nonselfdual pair, i.e. $[B]_\mathcal{F} = [\neg A]_\mathcal{F}$. Thus the ordering induced on the $\mathcal{F}$-degrees is *almost* a linear-order: this is the reason for which the principle $\mathsf{SLO}^\mathcal{F}$ is called "Semi-Linear Ordering Principle" for $\mathcal{F}$.

Lemma 2.1. *Let $\mathcal{F} \subseteq \mathcal{G}$ be two sets of functions from $\mathbb{R}$ to $\mathbb{R}$. Then $\mathsf{SLO}^\mathcal{F} \Rightarrow \mathsf{SLO}^\mathcal{G}$ and if we assume $\mathsf{SLO}^\mathcal{F}$*

$$\forall A, B \subseteq \mathbb{R}(A <_\mathcal{G} B \Rightarrow A <_\mathcal{F} B).$$

Proof. The first part is obvious, since $A \leq_\mathcal{F} B \Rightarrow A \leq_\mathcal{G} B$. For the second one, notice that $B \not\leq_\mathcal{F} A$ (otherwise $B \leq_\mathcal{G} A$). Since $\mathsf{SLO}^\mathcal{F} \Rightarrow \mathsf{SLO}^\mathcal{G}$, $A <_\mathcal{G} \neg B$: if $A \not\leq_\mathcal{F} B$ then $\neg B \leq_\mathcal{F} A$ by $\mathsf{SLO}^\mathcal{F}$, and hence $\neg B \leq_\mathcal{G} A$, a contradiction! $\square$

Recall also that if $\mathcal{F}$ is not too large and $\mathsf{SLO}^\mathcal{F}$ holds, then there is a uniform way to construct from a set $A \subseteq \mathbb{R}$ a new set $J_\mathcal{F}(A)$ which is $\mathcal{F}$-larger than A and $\neg A$ ($J_\mathcal{F}$ is also called *Solovay's jump operator*).

Lemma 2.2 (Solovay)**.** *Suppose that there is a surjection $j \colon \mathbb{R} \twoheadrightarrow \mathcal{F}$ and that $\mathsf{SLO}^\mathcal{F}$ holds. Then there is a map $J = J_\mathcal{F} \colon \mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R})$ such that*

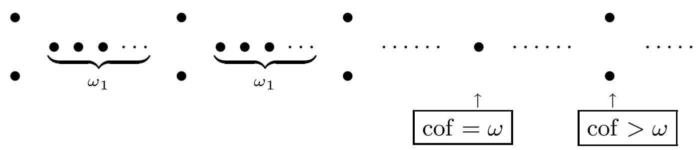
$$\forall A \subseteq \mathbb{R}(A <_\mathcal{F} J(A) \wedge \neg A <_\mathcal{F} J(A)).$$

2.3. **Wadge and Lipschitz degrees.** *We will assume $\mathsf{SLO}^\mathsf{L} + \neg\mathsf{FS} + \mathsf{DC}(\mathbb{R})$ for the rest of this Section* and state (without proof) some basic facts which will be useful for the next Sections. For more details see [14], [16] or [1].

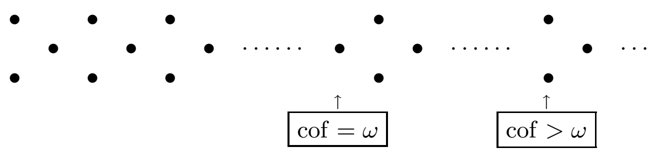
If $A_0, A_1, \ldots \subseteq \mathbb{R}$ are such that $\forall n \exists m > n(A_m \not\leq_\mathsf{L} A_n)$, then $\bigoplus_n A_n$ is L-selfdual and $[\bigoplus_n A_n]_\mathsf{L}$ is the least upper bound of the $[A_n]_\mathsf{L}$'s. In particular, $A \oplus \neg A$ is always L-selfdual and $[A \oplus \neg A]_\mathsf{L}$ is the least degree above $[A]_\mathsf{L}$ and $[\neg A]_\mathsf{L}$. Moreover, after a selfdual L-degree there is always another selfdual L-degree, and a limit L-degree is selfdual if and only if it is of countable cofinality, otherwise it is nonselfdual. Finally, the L-hierarchy is well-founded ($\|A\|_\mathsf{L}$ will denote the canonical rank of the

[2]For the sake of simplicity, all the terminology of this paragraph will be often applied in the obvious way to sets (rather than to $\mathcal{F}$-degrees).

set A with respect to $\leq_{\mathsf{L}}$) and its antichains have size at most 2. Therefore the Lipschitz hierarchy looks like this:



The description of the Wadge hierarchy can be obtained from the Lipschitz one using the Steel-Van Wesep Theorem (see e.g. Theorem 3.1 in [16]). A degree $[A]_{\mathsf{W}}$ is nonselfdual if and only if $[A]_{\mathsf{L}}$ is nonselfdual (and in this case $[A]_{\mathsf{W}} = [A]_{\mathsf{L}}$), while every selfdual degree $[A]_{\mathsf{W}}$ is exactly the union of an ω_1-block of consecutive Lipschitz degrees: therefore nonselfdual pairs and single selfdual degrees alternate in the W-hierarchy. Moreover, the W-hierarchy is well-founded and at limit levels of countable cofinality there is a single selfdual degree, while at limit levels of uncountable cofinality there is a nonselfdual pair. Hence the structure of the Wadge degrees looks like this:



Finally, recall also that the length of the Wadge hierarchy (as well as the length of the Lipschitz one) is $\Theta = \sup\{\alpha \mid \text{there is a surjective } f\colon \mathbb{R} \twoheadrightarrow \alpha\}$.

3. Sets of reductions

The idea behind the next definition is that we would like to use the functions from some $\mathcal{F} \subseteq {}^{\mathbb{R}}\mathbb{R}$ as *reductions*, and study the relation $\leq_{\mathcal{F}}$. In order to have a nontrivial structure, we must require $\mathcal{F}$ to be neither too small nor too large. For instance, we can not let $\mathcal{F}$ be the set of all constant functions plus the identity function (the latter must be adjoined to make $\leq_{\mathcal{F}}$ be a preorder), since in this case for different $A, B \subseteq \mathbb{R}$ we can have $A \leq_{\mathcal{F}} B$ only if $A = \mathbb{R}$ and $B \neq \emptyset$ or $A = \emptyset$ and $B \neq \mathbb{R}$ (in all the other cases we have $A \nleq_{\mathcal{F}} B$). Thus we get a degree-structure which is not very interesting, and the reason is basically that we have few functions to reduce one set to another: we can avoid this unpleasant situation requiring that $\mathcal{F}$ contains a very simple but sufficiently rich set of functions, such as the set L (note that this condition already implies $\mathrm{id} \in \mathcal{F}$). On the other hand, we want also to avoid that $\mathcal{F}$ contains too many functions. In fact, if for example we consider the set $\mathcal{F}$ of *all* the functions from $\mathbb{R}$ to $\mathbb{R}$ we have a lot of functions at our disposal, and we can reduce every set to any other (except for $\emptyset$ and $\mathbb{R}$), therefore we get a finite and trivial structure of $\mathcal{F}$-degrees (the same structure can be obtained considering any set $\mathcal{F}$ which contains all the two-valued functions). To avoid this situation we can require that $\mathcal{F}$ is not too large, i.e. that there is a surjection $j\colon \mathbb{R} \twoheadrightarrow \mathcal{F}$. All these considerations naturally lead to the following definition.

Definition 1. A set of functions $\mathcal{F} \subseteq {}^{\mathbb{R}}\mathbb{R}$ is a *set of reductions* if it is closed under composition, $\mathcal{F} \supseteq \mathsf{L}$ and there is a surjection $j\colon \mathbb{R} \twoheadrightarrow \mathcal{F}$.

Classical examples of sets of reductions are L, W, D_ξ, Bor and so on. It is interesting to note that this simple definition allows to describe almost completely the structure of the $\mathcal{F}$-degrees, as it is shown in the next Theorem.

Theorem 3.1 ($\mathsf{SLO}^{\mathsf{L}}$+$\neg\mathsf{FS}$+$\mathsf{DC}(\mathbb{R})$)**.** *Let $\mathcal{F} \subseteq {}^{\mathbb{R}}\mathbb{R}$ be a set of reductions. Then*

i) $\leq_{\mathcal{F}}$ is a well-founded preorder on $\mathscr{P}(\mathbb{R})$;

ii) *there is no $\leq_{\mathcal{F}}$-largest set and* $\mathrm{lh}(\leq_{\mathcal{F}}) = \Theta$*;*
iii) *anti-chains have length at most* 2 *and are of the form* $\{A, \neg A\}$ *for some set* A*;*
iv) $\mathbb{R} \not\leq_{\mathcal{F}} \neg\mathbb{R} = \emptyset$ *and if* $A \neq \emptyset, \mathbb{R}$ *then* $\emptyset, \mathbb{R} <_{\mathcal{F}} A$*;*
v) *if* $A \not\leq_{\mathcal{F}} \neg A$ *then* $A \oplus \neg A$ *is* $\mathcal{F}$*-selfudal and is the successor of both* A *and* $\neg A$*. In particular, after an* $\mathcal{F}$*-nonselfdual pair there is a single* $\mathcal{F}$*-selfdual degree;*
vi) *if* $A_0 <_{\mathcal{F}} A_1 <_{\mathcal{F}} \dots$ *is an* $\mathcal{F}$*-chain of subsets of* $\mathbb{R}$ *then* $\bigoplus_n A_n$ *is* $\mathcal{F}$*-selfdual and is the supremum of these sets. In particular if* $[A]_{\mathcal{F}}$ *is limit of countable cofinality then* $A \leq_{\mathcal{F}} \neg A$*;*
vii) *if* $A \not\leq_{\mathcal{F}} \neg A$ *and* $\mathcal{G} \subseteq \mathcal{F}$ *is another set of reductions then* $[A]_{\mathcal{F}} = [A]_{\mathcal{G}}$*. In particular,* $[A]_{\mathcal{F}} = [A]_{\mathsf{L}}$*.*

Proof. Part *iv)* follows from the fact that $\mathcal{F}$ contains all the constant functions (which are in L). For *ii)* note that since $\mathsf{L} \subseteq \mathcal{F}$ and we have $\mathsf{SLO}^{\mathsf{L}}$, we can also assume $\mathsf{SLO}^{\mathcal{F}}$ by Lemma 2.1. Thus, using the surjection $j : \mathbb{R} \twoheadrightarrow \mathcal{F}$, we can define the Solovay's jump operator $J_{\mathcal{F}}$ and use Lemma 2.2 to get the result with the standard argument (see e.g. Theorem 2.7 in [1]). Part *iii)* immediately follows from $\mathsf{SLO}^{\mathcal{F}}$. For part *vi)* and part *i)*, use the fact the each strict inequality with respect to $\mathcal{F}$-reductions can be converted in a strict inequality with respect to L-reductions by $\mathsf{SLO}^{\mathsf{L}}$ and Lemma 2.1: therefore the (proper) chain in part *vi)* can be converted in a (proper) chain with respect to L (and this gives that $\bigoplus_n A_n$ is selfdual and the supremum of the chain), while in part *i)* any descending $\mathcal{F}$-chain can be converted in a descending L-chain (the fact that the non-existence of a descending $\mathcal{F}$-chain implies well-foundness can be proved as in [4], using $\mathsf{DC}(\mathbb{R})$ and the surjection $j\colon \mathbb{R} \twoheadrightarrow \mathcal{F}$). For part *v)*, observe that it can not be the case that $A \leq_{\mathsf{L}} \neg A$ (otherwise we should have also $A \leq_{\mathcal{F}} \neg A$), and therefore $A \oplus \neg A$ is selfdual and is the immediate successor of A and $\neg A$. Finally, for part *vii)* we clearly have $[A]_{\mathcal{G}} \subseteq [A]_{\mathcal{F}}$. Towards a contradiction, assume that B is not $\mathcal{G}$-equivalent to A but $B \in [A]_{\mathcal{F}}$. Note that B can not be $\mathcal{G}$-equivalent to $\neg A$ (otherwise, $A \equiv_{\mathcal{F}} \neg A$), hence we have only two cases (we sistematically use part *v)*): if $B <_{\mathcal{G}} A$ then we would have that $B \leq_{\mathcal{G}} B \oplus \neg B \leq_{\mathcal{G}} A$. But then $A \equiv_{\mathcal{F}} B \oplus \neg B$ would be $\mathcal{F}$-selfdual, a contradiction! If $A <_{\mathcal{G}} B$ simply argue as above but replacing the role of $B \oplus \neg B$ with $A \oplus \neg A$ to get the same contradiction! □

It is useful to observe that in the previous Theorem part *iv)* is provable in ZF, while parts *ii)*-*iii)* are true under $\mathsf{SLO}^{\mathcal{F}}$ alone. There are essentially two points left open by Theorem 3.1 in the description of the hierarchy of the $\mathcal{F}$-degrees, namely:

Question 2: What happens at limit levels of uncountable cofinality?
Question 3: What happens after a selfdual degree?

In order to answer these two Questions, we first introduce some useful definitions and an important tool strictly related to the set of reductions $\mathcal{F}$.

From now on any pointclass closed under L-preimages will be called L-*pointclass* (note that, in particular, any boldface pointclass is an L-pointclass). More generally, if $\mathcal{F}$ is any *set of functions*, any pointclass closed under $\mathcal{F}$-preimages will be called $\mathcal{F}$-*pointclass* (and clearly if $\mathcal{F} \subseteq \mathcal{G}$ then every $\mathcal{G}$-pointclass is also an $\mathcal{F}$-pointclass).

Moreover if Γ is any pointclass we will say that $f\colon \mathbb{R} \to \mathbb{R}$ is a Γ-*function* if $f^{-1}(D) \in \Gamma$ for every $D \in \Gamma$ (this is clearly a natural generalization of the notion of $\boldsymbol{\Delta}^0_\xi$-function introduced before).

Definition 2. We will call *characteristic set of* $\mathcal{F} \subseteq {}^{\mathbb{R}}\mathbb{R}$ the pointclass

$$\Delta_{\mathcal{F}} = \{A \subseteq \mathbb{R} \mid A \leq_{\mathcal{F}} \mathbf{N}_{\langle 0 \rangle}\}.$$

Note if $\mathcal{F}$ is closed under composition $\Delta_{\mathcal{F}}$ is always an $\mathcal{F}$-pointclass, and that every $f \in \mathcal{F}$ is automatically a $\Delta_{\mathcal{F}}$-function (as we will see in some examples in

Section 6, the converse is not always true even if we assume that $\mathcal{F}$ is a set of reduction). Moreover, if $\mathsf{L} \subseteq \mathcal{F}$ then $\Delta_{\mathcal{F}}$ is selfdual and is also an L-pointclass.

Definition 3. A set of functions $\mathcal{F}$ is *saturated* if for every $f\colon \mathbb{R} \to \mathbb{R}$

$$f \in \mathcal{F} \iff f \text{ is a } \Delta_{\mathcal{F}}\text{-function.}$$

We will call the set $\mathrm{Sat}(\mathcal{F}) = \{f \in {}^{\mathbb{R}}\mathbb{R} \mid f \text{ is a } \Delta_{\mathcal{F}}\text{-function}\}$ the *saturation* of $\mathcal{F}$. Clearly if $\mathcal{F}$ is closed under composition $\mathcal{F} \subseteq \mathrm{Sat}(\mathcal{F})$, and $\mathcal{F} = \mathrm{Sat}(\mathcal{F})$ just in case $\mathcal{F}$ is saturated. Moreover if $\mathrm{id} \in \mathcal{F}$ then $\Delta_{\mathrm{Sat}(\mathcal{F})} = \Delta_{\mathcal{F}}$. In fact, $\Delta_{\mathcal{F}} \subseteq \Delta_{\mathrm{Sat}(\mathcal{F})}$ by the previous observation. Conversely, let $A \in \Delta_{\mathrm{Sat}(\mathcal{F})}$. By definition there is some $f \in \mathrm{Sat}(\mathcal{F})$ such that $A = f^{-1}(\mathbf{N}_{\langle 0\rangle})$, and since f is a $\Delta_{\mathcal{F}}$-function and $\mathbf{N}_{\langle 0\rangle} \in \Delta_{\mathcal{F}}$ we have also $A \in \Delta_{\mathcal{F}}$. Thus $\mathrm{Sat}(\mathcal{F})$ is a maximum (with respect to inclusion) among those sets of reductions $\mathcal{G}$ such that $\Delta_{\mathcal{G}} = \Delta_{\mathcal{F}}$.

Remark 3.2. Assume that $\mathcal{F}$ is a set of functions (but not necessarily a set of reductions). Assuming $\mathsf{SLO}^{\mathsf{L}}$ and that each $\mathbf{N}_s$ is in $\Delta_{\mathcal{F}}$, we get that if $\Delta_{\mathcal{F}}$ is *bounded* in the Lipschitz hierarchy (i.e. if there is some $B \subseteq \mathbb{R}$ such that $D \leq_{\mathsf{L}} B$ for every $D \in \Delta_{\mathcal{F}}$) then there is a surjection $j\colon \mathbb{R} \twoheadrightarrow \mathcal{F}$. In fact, we can take the bound B to be such that $B \not\leq_{\mathsf{W}} \neg B$ and $\mathbf{\Gamma} = \{A \subseteq \mathbb{R} \mid A \leq_{\mathsf{L}} B\}$ is a boldface pointclass which contains all the countable intersections of sets in $\Delta_{\mathcal{F}}$ (so that, in particular, we have that $\mathbf{\Gamma}$ admits a universal set $U \subseteq \mathbb{R} \times \mathbb{R}$ — see e.g. Theorem 3.1 in [1]). Since

$$(x, y) \in \mathrm{graph}(f) \iff \forall s \in {}^{<\omega}\omega (y \in \mathbf{N}_s \Rightarrow x \in f^{-1}(\mathbf{N}_s)),$$

we have that the graph of any $f \in \mathcal{F}$ is in $\mathbf{\Gamma}$. Now let f_0 be any fixed function in $\mathcal{F}$ and for every $x \in \mathbb{R}$ let $j(x) = f$ if $h^{-1}(U_x) = \mathrm{graph}(f)$, and $j(x) = f_0$ otherwise: it is not hard to check that j is the surjection required.

Conversely, assume that $\Delta_{\mathcal{F}}$ is unbounded (i.e. "cofinal") in the Lipschitz hierarchy and that there is a surjection $j\colon \mathbb{R} \twoheadrightarrow \mathcal{F}$. Let i be a surjection of $\mathbb{R}$ onto L and for every $x, y \in \mathbb{R}$ put $f_x = j(x)$ and $l_y = i(y)$. Now assume $\mathsf{SLO}^{\mathsf{L}} + \neg\mathsf{FS} + \mathsf{DC}(\mathbb{R})$ and define $s\colon \mathbb{R} \to \Theta\colon x \oplus y \mapsto \|l_y^{-1}(f_x^{-1}(\mathbf{N}_{\langle 0\rangle}))\|_{\mathsf{L}}$. It is easy to check that s is onto, contradicting the definition of Θ.

Therefore, under $\mathsf{SLO}^{\mathsf{L}} + \neg\mathsf{FS} + \mathsf{DC}(\mathbb{R})$ any set of functions $\mathcal{F}$ closed under composition and such that $\mathbf{N}_s \in \Delta_{\mathcal{F}}$ for every $s \in {}^{<\omega}\omega$ admits a surjection $j\colon \mathbb{R} \twoheadrightarrow \mathcal{F}$ if and only if $\Delta_{\mathcal{F}}$ is bounded in the Lipschitz (equivalently, Wadge) hierarchy. In particular, if $\mathcal{F} \supseteq \mathsf{L}$ and $\mathcal{F}$ satisfies the previous conditions, then it is a set of reductions if and only if $\Delta_{\mathcal{F}} \neq \mathscr{P}(\mathbb{R})$ (hence the unique saturated set of functions closed under composition and which contains L but is not a set of reductions is the set of *all* the functions from $\mathbb{R}$ to $\mathbb{R}$). This means that the condition of "smallness" on the sets of functions presented at page 5 could be reformulated as "$\Delta_{\mathcal{F}}$ bounded in the Lipschitz hierarchy" or simply "$\Delta_{\mathcal{F}} \neq \mathscr{P}(\mathbb{R})$".

Now we return to consider *sets of reductions*. It is clear that $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \Delta_{\mathcal{F}} \subseteq \Delta_{\mathcal{G}}$, but the converse is not always true, as we will see in some examples in Section 6. Nevertheless, there is a noteworthy situation in which $\Delta_{\mathcal{F}} \subseteq \Delta_{\mathcal{G}}$ implies $\mathcal{F} \subseteq \mathcal{G}$, namely when $\mathcal{G} = \mathsf{D}_\xi$ (for some nonzero $\xi < \omega_1$), although in this case we must also require that $\mathbf{N}_s \in \Delta_{\mathcal{F}}$ for every $s \in {}^{<\omega}\omega$ (we will see that this is always the case if $\mathcal{F}$ is a *Borel-amenable* set of reductions). First notice that $\Delta_{\mathsf{D}_\xi} = \mathbf{\Delta}^0_\xi$ (and $\Delta_{\mathsf{Bor}} = \mathbf{\Delta}^1_1$).

Proposition 3.3. *Let $\mathcal{F}$ be a set of reductions such that each $\mathbf{N}_s$ is in $\Delta_{\mathcal{F}}$, and let $\xi < \omega_1$ be a nonzero ordinal. If $\Delta_{\mathcal{F}} \subseteq \mathbf{\Delta}^0_\xi$ then $\mathcal{F} \subseteq \mathsf{D}_\xi$.*

Proof. Since $\mathsf{D}_\xi \subseteq \mathsf{D}_{\xi'}$ for $\xi < \xi'$, we may assume that ξ is least such that $\Delta_{\mathcal{F}} \subseteq \mathbf{\Delta}^0_\xi$. Suppose $\xi > 1$ and let $f \in \mathcal{F}$: by definition of D_ξ we must show that if A is $\mathbf{\Sigma}^0_\xi$,

then so is $f^{-1}(A)$. Choose $A_n \in \mathbf{\Pi}^0_{\nu_n}$ with $\nu_n < \xi$ such that $A = \bigcup_{n\in\omega} A_n$. By Borel-determinacy either $\mathbf{\Delta}^0_{\nu_n} \subseteq \Delta_{\mathcal{F}}$ or $\Delta_{\mathcal{F}} \subseteq \mathbf{\Delta}^0_{\nu_n}$, but the latter cannot hold by the minimality of ξ, hence $\mathbf{\Delta}^0_{\nu_n} \subsetneq \Delta_{\mathcal{F}}$ and therefore $\mathbf{\Pi}^0_{\nu_n} \subsetneq \Delta_{\mathcal{F}}$ for each n. As f is a $\Delta_{\mathcal{F}}$ function, $f^{-1}(A_n) \in \Delta_{\mathcal{F}} \subseteq \mathbf{\Delta}^0_\xi$, hence $f^{-1}(A) \in \mathbf{\Sigma}^0_\xi$ as required.

The case $\xi = 1$ is trivial, and it is left to the reader. □

Notice that the same result is trivially true if we consider Borel functions instead of $\mathbf{\Delta}^0_\xi$-functions.

4. Borel-amenability

A. Andretta and D. A. Martin proposed in [4] a notion of *amenable* set of functions essentially adjoining the following condition to Definition 1: *for every countable family $\{f_n \mid n \in \omega\} \subseteq \mathcal{F}$ we have that $\overline{\bigoplus}_n f_n \in \mathcal{F}$, where*

$$\overline{\bigoplus}_n f_n(x) = f_{x(0)}(x^-)$$

and $x^- = \langle x(n+1) \mid n \in \omega\rangle$. This condition can be recast in a different way.

Proposition 4.1. *The following are equivalent:*

i) if $\{f_k \mid k \in \omega\} \subseteq \mathcal{F}$ then $\overline{\bigoplus}_k f_k \in \mathcal{F}$;
ii) if $\{f_k \mid k \in \omega\} \subseteq \mathcal{F}$ then $\bigcup_k (f_k \restriction \mathbf{N}_{\langle k\rangle}) \in \mathcal{F}$ and $\mathsf{Lip} \subseteq \mathcal{F}$.

Proof. To see *ii)* $\Rightarrow$ *i)*, let $\{f_k \mid k \in \omega\} \subseteq \mathcal{F}$ and define $f_k^-(x) = (f_k(x))^-$ for every $x \in \mathbb{R}$ and $k \in \omega$: then clearly $f_k^- \in \mathcal{F}$ (since $\mathsf{Lip} \subseteq \mathcal{F}$) and $\overline{\bigoplus}_k f_k = \bigcup_k (f_k^- \restriction \mathbf{N}_{\langle k\rangle}) \in \mathcal{F}$. To prove *i)* $\Rightarrow$ *ii)*, let $\{f_k \mid k \in \omega\} \subseteq \mathcal{F}$ and for every $k \in \omega$ put

$$f_k^+ \colon \mathbb{R} \to \mathbb{R} \colon x \mapsto f_k(k^\frown x).$$

Since for every $k \in \omega$ the function $x \mapsto k^\frown x$ is in $\mathsf{L} \subseteq \mathcal{F}$, we get $f_k^+ \in \mathcal{F}$ and hence $\overline{\bigoplus}_k f_k^+ = \bigcup_k (f_k \restriction \mathbf{N}_{\langle k\rangle}) \in \mathcal{F}$. Let now $f \in \mathsf{Lip}$, and let $n \in \omega$ be smallest such that $f \in \mathsf{Lip}(2^n)$: we will prove by induction on n that $f \in \mathcal{F}$. If $n = 0$ then $f \in \mathsf{L} \subseteq \mathcal{F}$. Now assume that $\mathsf{Lip}(2^n) \subseteq \mathcal{F}$ and pick any $f \in \mathsf{Lip}(2^{n+1})$. For every $k \in \omega$ define $f_k \colon \mathbb{R} \to \mathbb{R} \colon x \mapsto f(k^\frown x)$. Since for every $x, y \in \mathbb{R}$

$$d(f_k(x), f_k(y)) = d(f(k^\frown x), f(k^\frown y)) \leq$$
$$\leq 2^{n+1} d(k^\frown x, k^\frown y) = 2^{n+1} \cdot \frac{1}{2} d(x,y) = 2^n d(x,y),$$

we have $\{f_k \mid k \in \omega\} \subseteq \mathsf{Lip}(2^n) \subseteq \mathcal{F}$, and thus $f = \overline{\bigoplus}_k f_k \in \mathcal{F}$ by our hypotheses. □

Roughly speaking, this condition of amenability says that ($\mathsf{Lip} \subseteq \mathcal{F}$ and) if we have a "simple" partition of $\mathbb{R}$, i.e. composed by the simplest (in the sense of $\leq_{\mathsf{L}}$) nontrivial sets (namely, sets in Δ_{L}: every Δ_{L}-partition is always refined by $\langle \mathbf{N}_{\langle k\rangle} \mid k \in \omega\rangle$, so our definition based only on sets of the form $\mathbf{N}_{\langle k\rangle}$ is not restrictive), and we use on each piece of the partition a function from $\mathcal{F}$ (as complex as we want), the resulting function is already in $\mathcal{F}$. But the simplest sets from "the point of view" of $\mathcal{F}$ are those in $\Delta_{\mathcal{F}}$, hence it seems quite natural to extend the definition of "amenable" to the following:

Definition 4. A set of reductions $\mathcal{F} \subseteq \mathsf{Bor}$ is *Borel-amenable* if:

(1) $\mathsf{Lip} \subseteq \mathcal{F}$;
(2) for every $\Delta_{\mathcal{F}}$-partition $\langle D_n \mid n \in \omega\rangle$ and every collection $\{f_n \mid n \in \omega\}$ of functions from $\mathcal{F}$ we have that

$$f = \bigcup_{n\in\omega} (f_n \restriction D_n) \in \mathcal{F}.$$

We will denote by BAR the set of all the Borel-amenable sets of reductions.

Remark 4.2. The condition $\mathcal{F} \subseteq \mathsf{Bor}$, as already observed, can be recast in an equivalent way by requiring that $\Delta_{\mathcal{F}} \subseteq \mathbf{\Delta}^1_1$. Note also that this condition already implies that there is a surjection $j \colon \mathbb{R} \twoheadrightarrow \mathcal{F}$ (as observed in Remark 3.2), while the condition $\mathsf{Lip} \subseteq \mathcal{F}$ implies that $\mathsf{L} \subseteq \mathcal{F}$: hence if a *set of functions* $\mathcal{F} \subseteq \mathsf{Bor}$ satisfies the two conditions in Definition 4 and is closed under composition, then it is automatically a Borel-amenable *set of reductions*.

Almost all the sets of functions one is willing to use as reductions, such as continuous functions, $\mathbf{\Delta}^0_\xi$-functions and Borel functions, are examples of Borel-amenable sets of reductions. In particular, by Proposition 4.1, a Borel-amenable set of reductions $\mathcal{F}$ is always closed under the operation $\overline{\bigoplus}_n$, and since $\mathbf{\Delta}^0_1$ is the smallest L-pointclass closed under $\bigoplus_n$, it is easy to check that $\mathbf{\Delta}^0_1 \subseteq \Delta_{\mathcal{F}}$ (hence, in particular, $\mathbf{N}_s \in \Delta_{\mathcal{F}}$ for every $s \in {}^{<\omega}\omega$). Moreover, if $\mathcal{F} \in \mathsf{BAR}$ then parts *ii)-vi)* of Theorem 3.1 are true under $\mathsf{SLO}^{\mathcal{F}}$ alone (see Lemma 3 of [4] for parts *v)* and *vi)*), and if $\mathcal{F}, \mathcal{G} \in \mathsf{BAR}$ then the first part of *vii)* is provable also under $\mathsf{SLO}^{\mathcal{G}}$. Finally, we can also establish a minimum among those Borel-amenable sets of reductions with the same characteristic set. In fact, given a Borel-amenable set of reductions $\mathcal{F}$, let $\mathcal{F}^{\mathsf{Lip}}$ be the collection of the functions locally in Lip on a $\Delta_{\mathcal{F}}$-partition. Then $\mathcal{F}^{\mathsf{Lip}} \subseteq \mathcal{F}$ since $\mathcal{F}$ must satisfy condition 2 of the definition of Borel-amenability and $\mathsf{Lip} \subseteq \mathcal{F}$. This implies also $\Delta_{\mathcal{F}^{\mathsf{Lip}}} \subseteq \Delta_{\mathcal{F}}$. Conversely, $\Delta_{\mathcal{F}} \subseteq \Delta_{\mathcal{F}^{\mathsf{Lip}}}$ since if $D \in \Delta_{\mathcal{F}}$ the function $f_D = (g_0 \restriction D) \cup (g_1 \restriction \neg D)$, where $g_0, g_1 \in \mathsf{L} \subseteq \mathcal{F}$ are the constant functions with value, respectively, $\vec{0}$ and $\vec{1}$, is in $\mathcal{F}^{\mathsf{Lip}}$ and reduces D to $\mathbf{N}_{\langle 0 \rangle}$. Thus if $\mathcal{G}$ is such that $\Delta_{\mathcal{G}} = \Delta_{\mathcal{F}}$ then $\mathcal{F}^{\mathsf{Lip}} = \mathcal{G}^{\mathsf{Lip}} \subseteq \mathcal{G}$. In particular, this implies that $\mathsf{D}^{\mathsf{Lip}}_1$ (the set of all the functions locally in Lip on a clopen partition) is a subset of *any* Borel-amenable set of reductions. A similar argument allow us to prove the following result.

Proposition 4.3. *Let $\mathcal{F}$ be a Borel-amenable set of reductions. Then either $\Delta_{\mathcal{F}} = \mathbf{\Delta}^1_1$ or there is some nonzero $\xi < \omega_1$ such that $\Delta_{\mathcal{F}} = \mathbf{\Delta}^0_\xi$.*

Proof. Assume that $\Delta_{\mathcal{F}} \subsetneq \mathbf{\Delta}^1_1$ and let $\xi < \omega_1$ be the smallest nonzero ordinal such that $\Delta_{\mathcal{F}} \subseteq \mathbf{\Delta}^0_\xi$. If $D \in \mathbf{\Delta}^0_\xi$, then there is some partition $\langle D_n \mid n \in \omega \rangle$ of $\mathbb{R}$ such that $D = \bigcup_{i \in I} D_i$ for some $I \subseteq \omega$ and $D_n \in \mathbf{\Pi}^0_{\mu_n}$ for some $\mu_n < \xi$ (see Theorem 4.2 in [11]). Since $\mathbf{\Delta}^0_\mu \subsetneq \Delta_{\mathcal{F}}$ for every $\mu < \xi$ (by minimality of ξ), we have $\{D_n \mid n \in \omega\} \subseteq \bigcup_{\mu < \xi} \mathbf{\Pi}^0_\mu \subseteq \Delta_{\mathcal{F}}$. Let g_0, g_1 be defined as above, and put $f_i = g_0$ if $i \in I$ and $f_i = g_1$ otherwise. By Borel-amenability, $f = \bigcup_{n \in \omega}(f_n \restriction D_n) \in \mathcal{F}$ and $f^{-1}(\mathbf{N}_{\langle 0 \rangle}) = D$, i.e. $D \in \Delta_{\mathcal{F}}$: therefore $\mathbf{\Delta}^0_\xi \subseteq \Delta_{\mathcal{F}}$. $\square$

Notice that this Proposition easily implies Proposition 3.3 (in the special case $\mathcal{F} \in \mathsf{BAR}$) and that $\Delta_{\mathcal{F}}$ is always an algebra of sets, i.e. that it is closed under complements, finite intersections and finite unions (this fact directly follows also from Lemma 4.4). Moreover, as a corollary one gets that either $\mathrm{Sat}(\mathcal{F}) = \mathsf{Bor}$ or $\mathrm{Sat}(\mathcal{F}) = \mathsf{D}_\xi$ for some nonzero $\xi < \omega_1$.

In the following couple of Lemmas we will always assume $\mathcal{F} \in \mathsf{BAR}$. They are analogous of Lemma 12 and, respectively, Lemma 13 and Proposition 18 of [4], and can be proved in almost the same way (here it is enough to use the second condition of Definition 4).

Lemma 4.4. *Let $D \subseteq D'$ be two sets in $\Delta_{\mathcal{F}}$. For every $A \subseteq \mathbb{R}$, if $A \cap D' \neq \mathbb{R}$ then $A \cap D \leq_{\mathcal{F}} A \cap D'$. In particular, if $D \in \Delta_{\mathcal{F}}$ and $A \neq \mathbb{R}$ then $A \cap D \leq_{\mathcal{F}} A$.*

Lemma 4.5. *Let $\langle D_n \mid n \in \omega \rangle$ be a $\Delta_{\mathcal{F}}$-partition of $\mathbb{R}$ and let $A \neq \mathbb{R}$.*

a) *If $C \subseteq \mathbb{R}$ and $A \cap D_n \leq_{\mathcal{F}} C$ for every $n \in \omega$ then $A \leq_{\mathcal{F}} C$.*
b) *Assume $\mathsf{SLO}^{\mathcal{F}}$. If $\forall n \in \omega (A \cap D_n <_{\mathcal{F}} A)$ then $A \leq_{\mathcal{F}} \neg A$. Moreover, if $D_n = \emptyset$ for all but finitely many n's then A is not limit.*
c) *Assume $\mathsf{SLO}^{\mathcal{F}}$. If $A \leq_{\mathcal{F}} \neg A$ and $[A]_{\mathcal{F}}$ is immediately above a nonselfdual pair $\{[C]_{\mathcal{F}}, [\neg C]_{\mathcal{F}}\}$ with $C \neq \emptyset, \mathbb{R}$, then there is $D \in \Delta_{\mathcal{F}}$ such that $A \cap D, A \cap \neg D <_{\mathcal{F}} A$.*

Definition 5. Let $\mathcal{F} \in \mathsf{BAR}$ and $A \subseteq \mathbb{R}$. We say that A has the *decomposition property with respect to* $\mathcal{F}$ if there is a $\Delta_{\mathcal{F}}$-partition $\langle D_n \mid n \in \omega \rangle$ of $\mathbb{R}$ such that $D_n \cap A <_{\mathcal{F}} A$ for every n.

Moreover, we will say that $\mathcal{F}$ has the *decomposition property* (**DP** for short) if every $\mathcal{F}$-selfdual set $A \notin \Delta_{\mathcal{F}}$ has the decomposition property with respect to $\mathcal{F}$.

Note that the property **DP** is essentially a converse to part *b)* of Lemma 4.5. The following Theorem is analogous to Corollary 17 of [4].

Theorem 4.6. *Let $\mathcal{F}$ be a Borel-amenable set of reductions with the* **DP**. *Then*

i) if $[A]_{\mathcal{F}}$ is limit of uncountable cofinality with respect to $\leq_{\mathcal{F}}$ then $A \nleq_{\mathcal{F}} \neg A$;
ii) assume $\mathsf{SLO}^{\mathcal{F}}$: then after a selfdual $\mathcal{F}$-degree there is an $\mathcal{F}$-nonselfdual pair.

Proof. i) Suppose that A is $\mathcal{F}$-limit of uncountable cofinality (hence, in particular, $A \notin \Delta_{\mathcal{F}}$) and assume towards a contradiction that $A \leq_{\mathcal{F}} \neg A$. Let $\langle D_n \mid n \in \omega \rangle$ be a $\Delta_{\mathcal{F}}$-partition of $\mathbb{R}$ such that $D_n \cap A <_{\mathcal{F}} A$ for each n (which exists since $\mathcal{F}$ has the **DP**). If $B \subseteq \mathbb{R}$ is such that $A \cap D_n \leq_{\mathcal{F}} B$ for every n then $A \leq_{\mathcal{F}} B$ by Lemma 4.5: hence A is the supremum of the family $\mathcal{A} = \{A \cap D_n \mid n \in \omega\}$ and therefore is of countable cofinality, a contradiction!

ii) It is enough to prove that if A and B are $\mathcal{F}$-selfdual and $A <_{\mathcal{F}} B$ (which in particular implies $B \notin \Delta_{\mathcal{F}}$) then B is not the successor of A. Using **DP**, let $\langle D_n \mid n \in \omega \rangle$ be a $\Delta_{\mathcal{F}}$-partition such that $D_n \cap B <_{\mathcal{F}} B$ for every n. If $D_n \cap B \leq_{\mathcal{F}} A$ for each $n \in \omega$, then $B \leq_{\mathcal{F}} A$ by Lemma 4.5, a contradiction! Thus there is some $n_0 \in \omega$ such that $D_{n_0} \cap B \nleq_{\mathcal{F}} A$: hence $\mathsf{SLO}^{\mathcal{F}}$ imples $\neg A \leq_{\mathcal{F}} D_{n_0} \cap B$, and since $A \leq_{\mathcal{F}} \neg A$ we get $A <_{\mathcal{F}} D_{n_0} \cap B <_{\mathcal{F}} B$. □

This proves that we can answer Question 1 and Question 2 for every $\mathcal{F} \in \mathsf{BAR}$ which satisfies **DP**. We will see in the next Section that, fortunately, *every* Borel-amenable set of reductions has this property, thus under $\mathsf{SLO}^{\mathsf{L}} + \neg\mathsf{FS} + \mathsf{DC}(\mathbb{R})$, we can completely determine the degree-structure induced by any reasonable (i.e. Borel-amenable) set of reductions. However we first want to go further and show that if $\mathcal{F}$ is as in the previous Proposition then the structure of the $\mathcal{F}$-degrees is completely determined by the set $\Delta_{\mathcal{F}}$. This is the reason for which the set $\Delta_{\mathcal{F}}$ has been called "characteristic set".

Theorem 4.7 ($\mathsf{SLO}^{\mathsf{L}}$)**.** *Let $\mathcal{F}, \mathcal{F}' \in \mathsf{BAR}$ be such that $\Delta_{\mathcal{F}} \subseteq \Delta_{\mathcal{F}'}$ and suppose that $\mathcal{F}$ has the* **DP**. *Then for every $A, B \subseteq \mathbb{R}$*

$$A \leq_{\mathcal{F}} B \Rightarrow A \leq_{\mathcal{F}'} B.$$

In particular, if $\mathcal{F} \in \mathsf{BAR}$ has the **DP** *and $\mathcal{F}' \subseteq \mathcal{F}$ is a Borel-amenable set of reductions such that $\Delta_{\mathcal{F}} = \Delta_{\mathcal{F}'}$, then for every $A, B \subseteq \mathbb{R}$*

$$A \leq_{\mathcal{F}} B \iff A \leq_{\mathcal{F}'} B.$$

Proof. We must take cases. If $A <_{\mathcal{F}} B$ then $A <_{\mathsf{L}} B$ by Lemma 2.1 and hence, in particular, $A \leq_{\mathcal{F}'} B$: thus we can assume $A \equiv_{\mathcal{F}} B$ for the other cases. If $A \nleq_{\mathcal{F}} \neg A$ then $[A]_{\mathcal{F}} = [A]_{\mathsf{L}}$ by part *vii)* of Theorem 3.1 and, since $B \in [A]_{\mathcal{F}}$ by our assumption, we have also $A \equiv_{\mathsf{L}} B$: thus $A \leq_{\mathcal{F}'} B$. If $A \in \Delta_{\mathcal{F}}$ then also $B \in \Delta_{\mathcal{F}}$, and hence $A \leq_{\mathcal{F}'} B$ since $\Delta_{\mathcal{F}} \subseteq \Delta_{\mathcal{F}'}$. Therefore it remains only to consider the case $A \equiv_{\mathcal{F}} \neg A \equiv_{\mathcal{F}} B \notin \Delta_{\mathcal{F}}$. Since $\mathcal{F}$ has the decomposition property, there is

some $\Delta_{\mathcal{F}}$-partition $\langle D_n \mid n \in \omega\rangle$ such that $D_n \cap A <_{\mathcal{F}} A \equiv_{\mathcal{F}} B$ for every n. In particular, using Lemma 2.1 and $\mathsf{SLO}^{\mathsf{L}}$, we have that $D_n \cap A <_{\mathsf{L}} B$ and hence, since $\mathcal{F}'$ is Borel-amenable and $\Delta_{\mathcal{F}} \subseteq \Delta_{\mathcal{F}'}$, $A \leq_{\mathcal{F}'} B$ by Lemma 4.5. □

Note that under the assumption $\mathcal{F}' \subseteq \mathcal{F}$ we can reprove the same result assuming only $\mathsf{SLO}^{\mathcal{F}'}$ instead of $\mathsf{SLO}^{\mathsf{L}}$. The previous Theorem allow also to compare different sets of reductions in term of the hierarchy of degrees induced by them: let us say that two sets of reductions $\mathcal{F}$ and $\mathcal{G}$ are *equivalent* ($\mathcal{F} \simeq \mathcal{G}$ in symbols) if they induce the same hierarchy of degrees, that is if for every $A, B \subseteq \mathbb{R}$ we have $A \leq_{\mathcal{F}} B$ if and only if $A \leq_{\mathcal{G}} B$. Then Theorem 4.7 implies that if $\mathcal{F}$ and $\mathcal{G}$ are two Borel-amenable sets of reductions (with the **DP**) we have

$$\mathcal{F} \simeq \mathcal{G} \iff \Delta_{\mathcal{F}} = \Delta_{\mathcal{G}}.$$

5. The Decomposition Property

In this Section we will prove that every Borel-amenable set of reductions $\mathcal{F}$ has the decomposition property, but we first need the following two Lemmas, which are refinements of Theorem 13.1 and Theorem 13.11 in [7]. Lemma 5.1 is a simple variation (for the Baire space endowed with the usual topology) of Theorem 22.18 of [7] (stated there, although in a slightly different form, as Exercise 22.20), while Lemma 5.2 follows from Lemma 5.1 by standard arguments. However we want to point out that these results could be generalized (with slightly different proofs) by considering *any* L-pointclass $\Delta \subseteq \mathbf{\Delta}^1_1(\mathbb{R}, \tau)$ — see [10]. This observation allows also to generalize Theorem 5.3 to almost all the sets of reductions $\mathcal{F} \subseteq \mathsf{Bor}$ (not only to the Borel-amenable ones), but we will not use this fact here.

Lemma 5.1. *Let d be the usual metric on $\mathbb{R}$, τ the topology induced by d, and ξ be any nonzero countable ordinal. Let Δ be either $\mathbf{\Delta}^0_\xi(\mathbb{R}, \tau)$ or $\mathbf{\Delta}^1_1(\mathbb{R}, \tau)$. For any family $\{D_n \mid n \in \omega\} \subseteq \Delta$ there is a metric d' on $\mathbb{R}$ such that*

- *i) $(\mathbb{R}, \tau')$ is Polish and zero-dimensional, where τ' is the topology induced by d';*
- *ii) τ' refines τ;*
- *iii) each D_n is τ'-clopen;*
- *iv) there is a countable clopen basis $\mathcal{B}'$ for τ' such that $\mathcal{B}' \subseteq \Delta$.*

Lemma 5.2. *Let d, τ, ξ and Δ be as in the previous Lemma . Moreover, let $\tau' \supseteq \tau$ be any zero-dimensional Polish topology on $\mathbb{R}$ which admits a countable clopen basis $\mathcal{B}' \subseteq \Delta$. For any Δ-function $f \colon \mathbb{R} \to \mathbb{R}$ there is a metric d_f on $\mathbb{R}$ such that*

- *i) $(\mathbb{R}, \tau_f)$ is Polish and zero-dimensional, where τ_f is the topology induced by d_f;*
- *ii) τ_f refines τ';*
- *iii) there is a countable clopen basis $\mathcal{B}_f$ for τ_f such that $\mathcal{B}_f \subseteq \Delta$;*
- *iv) $f \colon (\mathbb{R}, \tau_f) \to (\mathbb{R}, \tau')$ is continuous.*

Moreover d_f can be chosen in such a way that condition iv) can be strengthened to

- *iv') $f \colon (\mathbb{R}, \tau_f) \to (\mathbb{R}, \tau_f)$ is continuous.*

Now we are ready to prove the main Theorem of this Section, which sharpens the argument used to prove Theorem 16 in [4]. Since our proof closely follows the original one, we will only sketch it highlighting the modification that one has to adopt in this new context. Therefore the reader interested in a complete proof should keep a copy of [4] on hand and read the corresponding proofs parallel to one another.

Theorem 5.3 ($\neg\mathsf{FS}$)**.** *Let $\mathcal{F}$ be a Borel-amenable set of reductions. Assume that $A \leq_{\mathcal{F}} \neg A \notin \Delta_{\mathcal{F}}$. Then A has the decomposition property with respect to $\mathcal{F}$.*

Proof. Suppose towards a contradiction that for every $\Delta_{\mathcal{F}}$-partition $\langle D_n \mid n \in \omega \rangle$ of $\mathbb{R}$ there is some $n_0 \in \omega$ such that $D_{n_0} \cap A \equiv_{\mathcal{F}} A$. The next Claim is quite similar to Claim 16.1 in [4], but we will completely reprove it here in order to fill a little gap in the original proof.

Claim 5.3.1. Let $D \in \Delta_{\mathcal{F}}$ and assume $A \cap D \equiv_{\mathcal{F}} A$. Then there is some $f \in \mathcal{F}$ such that $\mathrm{range}(f) \subseteq D$ and

$$\forall x \in D(x \in A \cap D \iff f(x) \in \neg A \cap D).$$

Proof of the Claim. Note that $D = \emptyset$ and $D \subseteq A$ are forbidden since $D <_{\mathcal{F}} A$ (and $A \cap D = D$ would contradict $A \cap D \equiv_{\mathcal{F}} A$), while if $D = \mathbb{R}$ any $\mathcal{F}$-reduction of A to $\neg A$ will suffice. Thus we can assume $D \neq \emptyset, \mathbb{R}$ and $\neg A \cap D \neq \emptyset$. By Lemma 4.4 we have $\neg A \cap D \leq_{\mathcal{F}} \neg A \equiv_{\mathcal{F}} A \equiv_{\mathcal{F}} A \cap D$ ($\neg A$ is nonempty because it is selfdual). Let $h \in \mathcal{F}$ be such that $h^{-1}(A \cap D) = \neg A \cap D$ and choose some $y \in \neg A \cap D$. Now put

$$k(x) = \begin{cases} x & \text{if } x \in D \\ y & \text{if } x \notin D. \end{cases}$$

Note that $k \in \mathcal{F}$ (since $\mathcal{F}$ is Borel-amenable), and let $f = k \circ h$. Clearly $f \in \mathcal{F}$ and $\mathrm{range}(f) \subseteq D$. We will now prove that $x \in \neg A \cap D \iff f(x) \in A \cap D$ for every $x \in D$ (which easily implies the result). Let $x \in D$: if $x \in \neg A \cap D$ then $h(x) \in A \cap D \subseteq D$ and hence also $f(x) \in A \cap D$. Conversely, if $x \in A \cap D$ then $h(x) \in \neg A \cup \neg D$: if $h(x) \in D$ then $f(x) = h(x) \in \neg A \cap D$, otherwise $f(x) = y \in \neg A \cap D$ and in both cases we are done. $\square$ *Claim*

One must now construct the following sequences:

i) a sequence $\ldots \subseteq D_1 \subseteq D_0 = \mathbb{R}$ of sets in $\Delta_{\mathcal{F}}$ such that $A \cap D_n \equiv_{\mathcal{F}} A$ for every $n \in \omega$ (in particular, $D_n \neq \emptyset$);
ii) a sequence of functions $f_n \in \mathcal{F}$ as in the previous Claim, i.e. such that

$$\forall x \in D_n(x \in A \cap D_n \iff f_n(x) \in \neg A \cap D_n);$$

iii) a sequence of separable complete metrics d_n on $\mathbb{R}$ such that d_0 is the usual metric on $\mathbb{R}$, the topologies τ_n generated by the metrics d_n are all zero-dimensional, τ_{n+1} refines τ_n, D_n is clopen with respect to τ_n, every τ_n admits a countable clopen basis $\mathcal{B}_n \subseteq \Delta_{\mathcal{F}}$, the function $f_n \colon (\mathbb{R}, \tau_{n+1}) \to (\mathbb{R}, \tau_n)$ is continuous, and for every $m \leq n$ and every $x, y \in D_{n+1}$

$$d_m(g_m \circ \ldots \circ g_n(x), g_m \circ \ldots \circ g_n(y)) < 2^{-n}, \tag{$*$}$$

where each g_i is either $f_i \restriction D_{i+1}$ or the identity on D_{i+1}.

Then we can conclude our proof simply replacing the B_n's with the D_n's in the original proof (that is we can construct a flip-set from the sequences above: this gives the desired contradiction).

The construction of the required sequences is by induction on $n \in \omega$: set $D_0 = \mathbb{R}$, and let d_0 be the usual metric on $\mathbb{R}$ and $f_0 \in \mathcal{F}$ be any function witnessing $A \leq_{\mathcal{F}} \neg A$. Then suppose that D_m, $\mathcal{B}_m$, f_m and d_m have been defined for all $m \leq n$.

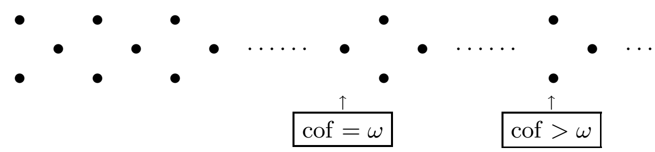
Claim 5.3.2. For each $m \leq n$ there is a $\Delta_{\mathcal{F}}$-partition $\langle C^i_m \mid i \in \omega \rangle$ of D_m such that d_m-$\mathrm{diam}(C^i_m) < 2^{-n}$ and C^i_m is τ_m-clopen for every $i \in \omega$.

Proof of the Claim. Since $\mathcal{B}_m \subseteq \Delta_{\mathcal{F}}$ is a countable basis for τ_m, we can clearly find a countably family $\{\hat{C}^i_m \mid i \in \omega\} \subseteq \Delta_{\mathcal{F}}$ such that $D_m \subseteq \bigcup_{i \in \omega} \hat{C}^i_m$ and d_m-$\mathrm{diam}(\hat{C}^i_m) < 2^{-n}$ for every $i \in \omega$. Now simply define $C^0_m = \hat{C}^0_m \cap D_m$ and $C^{i+1}_m = (\hat{C}^{i+1}_m \cap D_m) \setminus (\bigcup_{j \leq i} \hat{C}^j_m)$. Since $\Delta_{\mathcal{F}}$ is an algebra and each $\hat{C}^i_m$ is τ_m-clopen, $\langle C^i_m \mid i \in \omega \rangle$ is the required $\Delta_{\mathcal{F}}$-partition. $\square$ *Claim*

The inductive step can now be completed as in the original proof using the previous Claim and applying Lemma 5.2. $\square$

Notice that, as for the Borel case, the nonexistence of flip-sets is used in a "local way" in the proof of Theorem 5.3: in fact the flip-set obtained is the continuous preimage of A and therefore the proof only requires that there are no flip-sets W-reducible to A. Observe also that this kind or argument (which is based on relativizations of topologies) cannot be applied beyond the Borel context. In fact, if X and Y are Polish spaces, $f\colon X \to Y$ is a Borel function and $A \subseteq X$ is a Borel set such that $f \restriction A$ is injective, then also $f(A)$ is Borel (see Corollary 15.2 in [7]). Now suppose that τ and τ' are two Polish topologies on X such that $\boldsymbol{\Delta}^1_1(X,\tau) \subsetneq \boldsymbol{\Delta}^1_1(X,\tau')$ and let $A \in \boldsymbol{\Delta}^1_1(X,\tau') \setminus \boldsymbol{\Delta}^1_1(X,\tau)$. Applying the preceding result to $f = \mathrm{id}\colon (X,\tau') \to (X,\tau)$, we should have that $f(A) = \mathrm{id}(A) = A \in \boldsymbol{\Delta}^1_1(X,\tau)$, a contradiction! Therefore we can not refine the standard topology τ of $\mathbb{R}$ in order to make clopen (or even just Borel) a set which was not in $\boldsymbol{\Delta}^1_1(\mathbb{R},\tau)$ without losing the essential condition that the new space is still Polish[3]. Nevertheless it is possible to study the hierarchies of degrees induced by sets of reductions $\mathcal{F} \supsetneq \mathsf{Bor}$ using a different kind of argument — see the forthcoming [9].

Theorem 5.3 (together with Theorem 4.6) completes the description of the degree-structure induced by $\leq_{\mathcal{F}}$ when $\mathcal{F}$ is a Borel-amenable set of reductions, showing that the $\mathcal{F}$-structure looks like the structure of the Wadge degrees:



Recall also that BAR, in particular, contains almost all the cases already studied, namely continuous functions, $\boldsymbol{\Delta}^0_2$-functions and Borel functions: thus these results provide an alternative proof for the results about those degree-structures. Moreover, we highlight that the principle $\mathsf{SLO}^{\mathsf{L}}$ is needed only to prove the well-foundness of $\leq_{\mathcal{F}}$, since all the other results are provable under $\mathsf{SLO}^{\mathcal{F}} + \neg\mathsf{FS} + \mathsf{DC}(\mathbb{R})$.

We conclude this Section with the following Corollary which completely characterize the $\mathcal{F}$-selfdual degrees.

Corollary 5.4 ($\mathsf{SLO}^{\mathsf{L}}+\neg\mathsf{FS}+\mathsf{DC}(\mathbb{R})$)**.** *Let $\mathcal{F}$ be a Borel-amenable set of reductions and let $A \subseteq \mathbb{R}$ be such that $A \notin \Delta_{\mathcal{F}}$. Then the following are equivalent:*

i) *$A \leq_{\mathcal{F}} \neg A$;*
ii) *A has the decomposition property with respect to $\mathcal{F}$;*
iii) *if B is L-minimal in $[A]_{\mathcal{F}}$ then $B \leq_{\mathsf{L}} \neg B$ and B is either limit (of countable cofinality) or successor of a nonselfdual pair with respect to $\leq_{\mathsf{L}}$.*

Proof. That *i)* is equivalent to *ii)* follows directly from Theorem 5.3 and Lemma 4.5, and obviously *iii)* implies *i)*. It remains to prove that *i)* implies *iii)*. By Theorem 5.3 and Theorem 4.6, we must distinguish two cases: if A is limit with respect to $\leq_{\mathcal{F}}$, then there must be a countable chain $A_0 <_{\mathcal{F}} A_1 <_{\mathcal{F}} \dots$ such that A is the supremum of it: but in this case we get $A_0 <_{\mathsf{L}} A_1 <_{\mathsf{L}} \dots$ by $\mathsf{SLO}^{\mathsf{L}}$ and Lemma 2.1, and therefore it is easy to check that $\bigoplus_n A_n$ is L-selfdual, is limit in the L-hierarchy and is also L-minimal in $[A]_{\mathcal{F}}$. Similarly, if $[A]_{\mathcal{F}}$ is a successor degree then there must be some $C \subseteq \mathbb{R}$ such that $C \nleq_{\mathcal{F}} \neg C$ and $A \equiv_{\mathcal{F}} C \oplus \neg C$: in this case it is easy to check that $C \oplus \neg C$ is L-selfdual, is L-minimal in $[A]_{\mathcal{F}}$, and its L-degree is the successor (in the L-hierarchy) of the nonselfdual pair $\{[C]_{\mathsf{L}}, [\neg C]_{\mathsf{L}}\}$. $\square$

[3]The author would like to thank A. Marcone for suggesting the present argument which considerably simplify a previous proof of this fact.

6. The structure of BAR

We now want to study the structure $\langle \mathsf{BAR}, \subseteq \rangle$. Clearly, as already observed in the previous Sections, $\mathsf{D}_1^{\mathsf{Lip}}$ and Bor are, respectively, the minimum and the maximum of this structure. Let now $\emptyset \neq \mathscr{B} \subseteq \mathsf{BAR}$ and put $\bigwedge \mathscr{B} = \bigcap \mathscr{B}$. Then $\bigwedge \mathscr{B} \in \mathsf{BAR}$ and, by the properties of the intersection, $\bigwedge \mathscr{B}$ is the infimum for $\mathscr{B}$ (with respect to inclusion). Moreover, $\Delta_{\bigwedge \mathscr{B}} = \bigcap_{\mathcal{F} \in \mathscr{B}} \Delta_{\mathcal{F}}$. For one direction $\Delta_{\bigwedge \mathscr{B}} \subseteq \bigcap_{\mathcal{F} \in \mathscr{B}} \Delta_{\mathcal{F}}$ by definition: for the converse, let $D \in \bigcap_{\mathcal{F} \in \mathscr{B}} \Delta_{\mathcal{F}}$ and let $g_0, g_1 \in \mathsf{L}$ be the constant fuctions with value, respectively, $\vec{0}$ and $\vec{1}$. By Borel-amenability, $f = (g_0 \restriction D) \cup (g_1 \restriction \neg D) \in \mathcal{F}$ for every $\mathcal{F} \in \mathscr{B}$, and $f^{-1}(\mathbf{N}_{\langle 0 \rangle}) = D$: hence $D \in \Delta_{\bigwedge \mathscr{B}}$. In particular, by Proposition 4.3 we have $\Delta_{\bigwedge \mathscr{B}} = \boldsymbol{\Delta}^0_\xi$, where $\xi = \min\{\mu \mid \boldsymbol{\Delta}^0_\mu = \Delta_{\mathcal{F}} \text{ for some } \mathcal{F} \in \mathscr{B}\}$.

Conversely, let $\mathscr{C}_{\mathscr{B}} = \{\mathcal{G} \in \mathsf{BAR} \mid \mathcal{F} \subseteq \mathcal{G} \text{ for every } \mathcal{F} \in \mathscr{B}\}$: clearly $\mathscr{C}_{\mathscr{B}} \neq \emptyset$ (since $\mathsf{Bor} \in \mathscr{C}_{\mathscr{B}}$), thus we can define $\bigvee \mathscr{B} = \bigwedge \mathscr{C}_{\mathscr{B}} = \bigcap \mathscr{C}_{\mathscr{B}}$. Obviously $\bigvee \mathscr{B} \in \mathsf{BAR}$, and if $\mathcal{G} \in \mathsf{BAR}$ is such that $\mathcal{F} \subseteq \mathcal{G}$ for every $\mathcal{F} \in \mathscr{B}$ then $\mathcal{G} \in \mathscr{C}_{\mathscr{B}}$ by definition: hence $\bigvee \mathscr{B} \subseteq \mathcal{G}$ and $\bigvee \mathscr{B}$ is the supremum for $\mathscr{B}$ with respect to inclusion. Thus we have proved that $\langle \mathsf{BAR}, \subseteq \rangle$ is a complete lattice with minimum and maximum. Note however that, contrarily to $\bigwedge \mathscr{B}$, the supremum $\bigvee \mathscr{B}$ has been defined in an undirected way and not starting from the elements of $\mathscr{B}$. To give a direct construction of $\bigvee \mathscr{B}$, first consider the map $*$ which sends a generic set of reductions $\mathsf{Lip} \subseteq \mathcal{F} \subseteq \mathsf{Bor}$ such that $\Delta_{\mathcal{F}}$ *is closed under finite intersections* (i.e. such that $\Delta_{\mathcal{F}}$ is an algebra) to the set

$$\mathcal{F}^* = \Big\{ \bigcup\nolimits_{n \in \omega} (f_n \restriction D_n) \mid f_n \in \mathcal{F} \text{ for every } n \in \omega \text{ and } \langle D_n \mid n \in \omega \rangle \text{ is a } \Delta_{\mathcal{F}}\text{-partition of } \mathbb{R} \Big\}.$$

It is easy to check that e.g. $\mathsf{Lip}^* = \mathsf{D}_1^{\mathsf{Lip}}$.

Theorem 6.1. *The map $*$ is a surjection on* BAR *such that:*

i) *$*$ is the identity on* BAR*, i.e. if $\mathcal{F} \in \mathsf{BAR}$ then $\mathcal{F}^* = \mathcal{F}$;*

ii) *$\mathcal{F}^*$ is the minimal Borel-amenable set of reductions (with respect to inclusion) which contains $\mathcal{F}$.*

Proof. Part *i)* is obvious, while for part *ii)* it is enough to observe that if $\mathcal{G}$ is any set of reductions which contains $\mathcal{F}$ then $\Delta_{\mathcal{F}} \subseteq \Delta_{\mathcal{G}}$: hence if $\mathcal{G}$ satisfies the second condition in Definition 4 then it must contain all the functions from $\mathcal{F}^*$. Since if $\mathsf{Lip} \subseteq \mathcal{F} \subseteq \mathsf{Bor}$ then also $\mathsf{Lip} \subseteq \mathcal{F}^* \subseteq \mathsf{Bor}$ (as $\mathcal{F} \subseteq \mathcal{F}^*$), it remains only to show that if $\Delta_{\mathcal{F}}$ is closed under finite intersection then $\mathcal{F}^*$ is closed under composition and satisfies the second condition in Definition 4. Let $f = \bigcup_{n \in \omega} (f_n \restriction D_n)$ and $g = \bigcup_{k \in \omega} (g_k \restriction C_k)$ be two functions from $\mathcal{F}^*$, and for every $n, k \in \omega$ put $D_{n,k} = f_n^{-1}(C_k) \cap D_n$. Since $f_n \in \mathcal{F}$ is a $\Delta_{\mathcal{F}}$-function and $\Delta_{\mathcal{F}}$ is closed under finite intersections we have that $\langle D_{n,k} \mid n, k \in \omega \rangle$ is a $\Delta_{\mathcal{F}}$-partition of $\mathbb{R}$. Moreover $g_k \circ f_n \in \mathcal{F}$ and $g \circ f = \bigcup_{n,k \in \omega} (g_k \circ f_n \restriction D_{n,k})$, hence $g \circ f \in \mathcal{F}^*$ by definition.

Let now $\langle D_n \mid n \in \omega \rangle$ be a $\Delta_{\mathcal{F}^*}$-partition of $\mathbb{R}$: we claim that it admits a refinement to a $\Delta_{\mathcal{F}}$-partition. In fact, fix any $n \in \omega$ and let $g = \bigcup_{k \in \omega} (g_k \restriction C_k) \in \mathcal{F}^*$ be a reduction of D_n to $\mathbf{N}_{\langle 0 \rangle}$: then the sets $g_k^{-1}(\mathbf{N}_{\langle 0 \rangle}) \cap C_k$ form a $\Delta_{\mathcal{F}}$-partition of D_n. Thus we can safely assume that each D_n is in $\Delta_{\mathcal{F}}$. Let now $\{f_n \mid n \in \omega\} \subseteq \mathcal{F}^*$, $\langle D'_{n,k} \mid k \in \omega \rangle$ be a $\Delta_{\mathcal{F}}$-partition of $\mathbb{R}$, and $\{f_{n,k} \mid k \in \omega\} \subseteq \mathcal{F}$ be such that $f_n = \bigcup_{k \in \omega} (f_{n,k} \restriction D'_{n,k})$ for every n. Clearly the sets $D_{n,k} = D_n \cap D'_{n,k}$ form a $\Delta_{\mathcal{F}}$-partition of $\mathbb{R}$: hence the function

$$f = \bigcup_{n \in \omega} (f_n \restriction D_n) = \bigcup_{n,k \in \omega} (f_{n,k} \restriction D_{n,k})$$

is in $\mathcal{F}^*$ by definition, and $\mathcal{F}^*$ satisfies the second condition of Definition 4. $\square$

Let now $\hat{\mathscr{B}}$ be the closure under composition of $\bigcup \mathscr{B}$, and let $\bigvee \mathscr{B}$ be obtained applying the map $*$ to $\hat{\mathscr{B}}$, i.e. $\bigvee \mathscr{B} = (\hat{\mathscr{B}})^*$. It is not hard to check that $\hat{\mathscr{B}}$ is a set of reductions such that $\mathsf{Lip} \subseteq \hat{\mathscr{B}} \subseteq \mathsf{Bor}$, and that $\Delta_{\hat{\mathscr{B}}} = \bigcup_{\mathcal{F} \in \mathscr{B}} \Delta_{\mathcal{F}}$ (to see this use the fact that every function in $\hat{\mathscr{B}}$ is the composition of a *finite* number of functions from $\bigcup \mathscr{B}$): hence $\Delta_{\hat{\mathscr{B}}}$ is an algebra and $\bigvee \mathscr{B} \in \mathsf{BAR}$ by Theorem 6.1. Moreover, if $\mathcal{G} \in \mathsf{BAR}$ is such that $\mathcal{F} \subseteq \mathcal{G}$ for every $\mathcal{F} \in \mathscr{B}$, then $\hat{\mathscr{B}} \subseteq \mathcal{G}$ and thus $\bigvee \mathscr{B} \subseteq \mathcal{G}$ by Theorem 6.1 again. Therefore $\bigvee \mathscr{B}$ is the supremum of $\mathscr{B}$ with respect to inclusion. Contrarily to the infimum case, one can still prove that $\Delta_{\bigvee \mathscr{B}} \supseteq \bigcup_{\mathcal{F} \in \mathscr{B}} \Delta_{\mathcal{F}}$, but in some cases the other inclusion can fail. In fact, given e.g. $\mathscr{B} = \{\mathsf{D}_m \mid m \in \omega\}$, we have that $\bigvee \mathscr{B}$ is formed by those functions f which are in $\bigcup \mathscr{B}$ on a $\boldsymbol{\Delta}^0_\omega$-partition (a collection which is different from D_ω, see later in this Section): thus

$$\bigcup_{\mathcal{F} \in \mathscr{B}} \Delta_{\mathcal{F}} = \bigcup_{n \in \omega} \boldsymbol{\Delta}^0_n \subsetneq \boldsymbol{\Delta}^0_\omega = \Delta_{\bigvee \mathscr{B}}.$$

However it is not hard to see that $\Delta_{\bigvee \mathscr{B}} = \boldsymbol{\Delta}^0_\xi$, where $\xi = \sup\{\mu \mid \boldsymbol{\Delta}^0_\mu = \Delta_{\mathcal{F}}$ for some $\mathcal{F} \in \mathscr{B}\}$. Therefore we have $\Delta_{\bigvee \mathscr{B}} = \bigcup_{\mathcal{F} \in \mathscr{B}} \Delta_{\mathcal{F}}$ if and only if there is some $\mathcal{F} \in \mathscr{B}$ such that $\Delta_{\mathcal{F}} = \boldsymbol{\Delta}^0_\xi = \Delta_{\bigvee \mathscr{B}}$.

Put now $\mathcal{F} \equiv \mathcal{G}$ just in case $\Delta_{\mathcal{F}} = \Delta_{\mathcal{G}}$. If we assume $\mathsf{SLO}^\mathsf{L} + \neg\mathsf{FS} + \mathsf{DC}(\mathbb{R})$ and $\mathcal{F}, \mathcal{G} \in \mathsf{BAR}$, then $\mathcal{F} \equiv \mathcal{G}$ if and only if $\mathcal{F} \simeq \mathcal{G}$ (by Theorem 5.3 and Theorem 4.7), hence it is quite natural to consider the quotient $\mathsf{BAR}/_\equiv$ together with the relation $\preceq$ defined by

$$[\mathcal{F}]_\equiv \preceq [\mathcal{G}]_\equiv \iff \Delta_{\mathcal{F}} \subseteq \Delta_{\mathcal{G}}$$

(again, assuming $\mathsf{SLO}^\mathsf{L} + \neg\mathsf{FS} + \mathsf{DC}(\mathbb{R})$, we have $[\mathcal{F}]_\equiv \preceq [\mathcal{G}]_\equiv$ if and only if $A \leq_{\mathcal{F}} B \Rightarrow A \leq_{\mathcal{G}} B$ for every $A, B \subseteq \mathbb{R}$). It follows from Proposition 4.3 that this structure is a well-founded linear order of length $\omega_1 + 1$. Each equivalence class is of the form $\{\mathcal{F} \in \mathsf{BAR} \mid \Delta_{\mathcal{F}} = \boldsymbol{\Delta}^0_\xi\}$ for some $1 \leq \xi \leq \omega_1$ (similarly to the case of the set of functions D_{ω_1}, for notational simplicity we put $\boldsymbol{\Delta}^0_{\omega_1} = \boldsymbol{\Delta}^1_1$), and for this reason we will say that $\mathcal{F} \in \mathsf{BAR}$ is *of level* ξ if $\Delta_{\mathcal{F}} = \boldsymbol{\Delta}^0_\xi$. Moreover, if we consider a single equivalence class endowed with the inclusion relation, arguing as before we get again a complete lattice with minimum and maximum (here the minimum and the maximum are, respectively, $\mathsf{D}^{\mathsf{Lip}}_\xi$ and D_ξ, where $1 \leq \xi \leq \omega_1$ is the level of any of the sets of reductions in the equivalence class considered).

We now want to give some examples of (different) Borel-amenable sets of reductions, showing at once that each level of BAR contains more than one element and that there are $\mathcal{F} \subsetneq \mathcal{G} \in \mathsf{BAR}$ such that $\Delta_{\mathcal{F}} = \Delta_{\mathcal{G}}$ (so that, in particular, $\mathcal{F} \neq \mathrm{Sat}(\mathcal{F})$). We extend the notation introduced on page 9.

Definition 6. Let $\mathcal{F}, \mathcal{G}$ be two sets of reductions. We will denote by $\mathcal{F}^{\mathcal{G}}$ the set of all the functions which are *in* $\mathcal{G}$ *on a* $\Delta_{\mathcal{F}}$*-partition*. In particular, for any nonzero ordinal $\xi \leq \omega_1$, we will denote by $\mathsf{D}^\mathsf{W}_\xi$ the set of all the functions which are *continuous on a* $\boldsymbol{\Delta}^0_\xi$*-partition*.

Remark 6.2. One must be cautious and pay attention to the definition of $\mathsf{D}^\mathsf{W}_\xi$, since it must be distincted from the set

$$\tilde{\mathsf{D}}^\mathsf{W}_\xi = \{f \in {}^\mathbb{R}\mathbb{R} \mid \text{there is a } \boldsymbol{\Delta}^0_\xi\text{-partition } \langle D_n \mid n \in \omega\rangle \text{ of } \mathbb{R}$$
$$\text{such that } f \restriction D_n \text{ is continuous for every } n\}.$$

Clearly $\mathsf{D}^\mathsf{W}_\xi \subseteq \tilde{\mathsf{D}}^\mathsf{W}_\xi$ for every $\xi \leq \omega_1$, and if $\xi \leq 2$ then we have also $\mathsf{D}^\mathsf{W}_\xi = \tilde{\mathsf{D}}^\mathsf{W}_\xi$. But if $\xi > 2$ then $\mathsf{D}^\mathsf{W}_\xi \subsetneq \tilde{\mathsf{D}}^\mathsf{W}_\xi$. To see this, put $D = \{x \in \mathbb{R} \mid \forall n \exists m > n(x(m) \neq$

$0)\}$. Clearly D and $\neg D$ form a $\mathbf{\Delta}^0_3$-partition of $\mathbb{R}$. For every $x \in \mathbb{R} \cup {}^{<\omega}\omega$ let $N_x = \{n < \mathrm{lh}(x) \mid x(n) \neq 0\}$ and define the function $f \colon \mathbb{R} \to \mathbb{R}$ by letting $f(x) = \langle x(n) \mid n \in N_x \rangle$ if $x \in D$ and $f(x) = \vec{0}$ otherwise. It is not hard to check that $f \restriction D$ and $f \restriction \neg D$ are continuous, thus $f \in \tilde{\mathsf{D}}^{\mathsf{W}}_\xi$ for every $\xi > 2$. Nevertheless one can prove that there is no $\xi \leq \omega_1$ such that $f \in \mathsf{D}^{\mathsf{W}}_\xi$. This is a consequence of the following Claim.

Claim 6.2.1. Let $\langle C_n \mid n \in \omega \rangle$ be *any* partition of $\mathbb{R}$ and $\{f_n \mid n \in \omega\} \subseteq {}^{\mathbb{R}}\mathbb{R}$ be such that $f \restriction C_n = f_n \restriction C_n$. Then f_{n_0} is not continuous for some $n_0 \in \omega$.

Proof of the Claim. Since the C_n's cover $\mathbb{R}$, by the Baire Category Theorem there must be some $n_0 \in \omega$ such that C_{n_0} is not nowhere dense, i.e. such that $\mathbf{N}_s \subseteq \mathrm{Cl}(C_{n_0})$ for some $s \in {}^{<\omega}\omega$. Observe that for every $A \subseteq \mathbb{R}$ and every continuous function $g \colon \mathbb{R} \to \mathbb{R}$, if $g \restriction A \cap D = f \restriction A \cap D$ then $g \restriction \mathrm{Cl}(A) \cap D = f \restriction \mathrm{Cl}(A) \cap D$ (by the continuity of f and g on D). Now assume, towards a contradiction, that f_{n_0} is continuous: then $f_{n_0} \restriction \mathbf{N}_s \cap D = f \restriction \mathbf{N}_s \cap D$. For every $k \in \omega$ put $x_k = s^\frown 0^{(k)\frown} \vec{1} \in \mathbf{N}_s \cap D$ and $y_k = s^\frown 0^{(k)\frown} \vec{2} \in \mathbf{N}_s \cap D$, and check that $f_{n_0}(x_k) = f(x_k) = t^\frown \vec{1}$ and $f_{n_0}(y_k) = f(y_k) = t^\frown \vec{2}$ for every $k \in \omega$, where $t = \langle s(n) \mid n \in N_s \rangle$. Since $x_k \to s^\frown \vec{0}$ and f_{n_0} is continuous on the whole $\mathbb{R}$, we must have $f_{n_0}(s^\frown \vec{0}) = t^\frown \vec{1}$. Similarly, since $y_k \to s^\frown \vec{0}$, by continuity of f_{n_0} again we should have $f_{n_0}(s^\frown \vec{0}) = t^\frown \vec{2} \neq t^\frown \vec{1}$, a contradiction! Thus f_{n_0} can not be continuous and we are done. $\square$ *Claim*

Observe now that $\mathsf{D}_1 = \mathsf{D}^{\mathsf{W}}_1$ and that, in particular, $\mathsf{D}^{\mathsf{W}}_\xi \subseteq \mathsf{D}_\xi$ for any nonzero $\xi \leq \omega_1$. By a remarkable Theorem of Jayne and Rogers (Theorem 5 in [6]) we have that $\mathsf{D}_2 = \mathsf{D}^{\mathsf{W}}_2$, and as an obvious corollary one gets also $\mathsf{D}_2 \simeq \mathsf{D}^{\mathsf{W}}_2$. The Jayne-Rogers Theorem (and its mentioned corollary) were used in [3] to observe that the so-called *backtrack functions* are exactly (and thus give the same hierarchy of degrees as) the functions in D_2: this allowed to use all the combinatorics arising from the backtrack game (for a definition of this game see [15] or [3]) for the study of the D_2-hierarchy, and thus it seems desirable to find some extension of the Jayne-Rogers Theorem in order to simplify the study of $\leq_{\mathsf{D}_\xi}$ when $2 < \xi < \omega_1$ (this problem was first posed by Andretta in his [1]). The first obvious generalization is the statement $\mathsf{D}^{\mathsf{W}}_\xi = \mathsf{D}_\xi$, but this immediately fails for every $2 < \xi \leq \omega_1$ by the counter-example given in Remark 6.2. A slightly weaker generalization (which is not in contrast with this Remark) leads to the following Conjecture.

Conjecture 1. Let $\xi < \omega_1$ be any nonzero ordinal. Then $\mathsf{D}_\xi = \tilde{\mathsf{D}}^{\mathsf{W}}_\xi$.

Unfortunately, as Andretta already observed in [1], this Conjecture is not true for any $\xi \geq \omega$. In fact there is a function $P \colon \mathbb{R} \to \mathbb{R}$ (called Pawlikowski function[4]) which is of Baire class 1 (hence it is also in D_ω) but has the property that for *any* countable partition $\langle A_n \mid n \in \omega \rangle$ of $\mathbb{R}$ there is some $n_0 \in \omega$ such that $P \restriction A_{n_0}$ is not continuous (see Lemma 5.4 in [5]): this means that $\tilde{\mathsf{D}}^{\mathsf{W}}_\xi \subsetneq \mathsf{D}_\xi$ for every $\xi \geq \omega$. Since both $\mathsf{D}^{\mathsf{W}}_\xi$ and $\tilde{\mathsf{D}}^{\mathsf{W}}_\xi$ are Borel-amenable, all these observations show that we have at least two (respectively, three) Borel-amenable sets of reductions of level ξ for $\xi > 2$ (respectively, $\xi \geq \omega$). Nevertheless note that the counter-example P does not allow to prove the failure of Conjecture 1 for finite levels since P turns out to be a "proper" $\mathbf{\Delta}^0_\omega$-function, i.e. $P \notin \mathsf{D}_n$ for any $n \in \omega$ (this fact will be explicitly proved in the forthcoming [8]): hence it remains an open problem to determine if Conjecture 1 holds when $2 < \xi < \omega$.

[4]In [5] the Pawlikowski function was defined on a space which is homeomorphic to the Cantor space ${}^{\omega}2$, but it is clear that it can be extended to a function P defined on $\mathbb{R}$ without losing the various properties of the original function (except for injectivity, which is not needed here).

One could think that Conjecture 1 fails in the general case because is too strong, and that the Jayne-Rogers Theorem could admit a weaker generalization which holds for every $\xi < \omega_1$. This objection suggests to formulate the following Conjecture, in which we require that for any $\mathbf{\Delta}^0_\xi$-function there must be some $\mathbf{\Delta}^0_\xi$-partition $\langle A_n \mid n \in \omega \rangle$ of $\mathbb{R}$ such that $f \restriction A_n$ is only "simpler" then f (instead of continuous).

Conjecture 2. Let $\xi < \omega_1$ be a nonzero ordinal. Then $f \colon \mathbb{R} \to \mathbb{R}$ is a $\mathbf{\Delta}^0_\xi$-function if and only if there is a $\mathbf{\Delta}^0_\xi$-partition $\langle A_n \mid n \in \omega \rangle$ of $\mathbb{R}$ such that, for every $n \in \omega$, $f \restriction A_n$ is $\mathbf{\Delta}^0_{\mu_n}$-function (for some $\mu_n < \xi$), i.e. $f^{-1}(D)$ is in $\mathbf{\Delta}^0_{\mu_n}$ *relatively to* A_n whenever $D \in \mathbf{\Delta}^0_{\mu_n}$.

One direction is trivial, but also this Conjecture fails for $\xi = \omega$ if we assume $\mathsf{DC}(\mathbb{R})$. The proof of this fact heavily rely on a deep result of Solecki which will be used to prove Proposition 6.3. Let X_1, Y_1, X_2 and Y_2 be separable metric spaces and pick any $f \colon X_1 \to Y_1$ and $g \colon X_2 \to Y_2$: then we say that f *is contained in* g ($f \sqsubseteq g$ in symbols) if and only if there are embeddings $\varphi \colon X_1 \to X_2$ and $\psi \colon f(X_1) \subseteq Y_1 \to Y_2$ such that $\psi \circ f = g \circ \varphi$. This notion of containment between functions allows to bound the complexity of a function by showing that it is contained in another function of known complexity. In fact it is straightforward to check that if f and g are as above we have that if $f \sqsubseteq g$ and g is a $\mathbf{\Delta}^0_\xi$-function (for some nonzero $\xi < \omega_1$) then also f is a $\mathbf{\Delta}^0_\xi$-function (similarly, if g is of Baire class ξ then also f is of Baire class ξ), and conversely if f is *not* a $\mathbf{\Delta}^0_\xi$-function (or a Baire class ξ function) then also g is *not* a $\mathbf{\Delta}^0_\xi$-function (or a Baire class ξ function).

Proposition 6.3 (ZFC)**.** *Let X be a Polish space and Y be a separable metric space. For any Baire class 1 function $f \colon X \to Y$ either there is some countable partition $\langle X_n \mid n \in \omega \rangle$ of X such that $f \restriction X_n$ is continuous for every $n \in \omega$, or else for every Borel partition (equivalently, $\mathbf{\Sigma}^1_1$-partition) $\langle A_n \mid n \in \omega \rangle$ of X there is some $k \in \omega$ such that $P \sqsubseteq f \restriction A_k$.*

Proof. Assume that the first alternative does not hold. Since each A_n is an analytic subset of a Polish space, by definition it is also Souslin and hence we can apply Solecki's Theorem 4.1 of [12] to $f \restriction A_n$. But by our assumption it can not be the case that the first alternative of Solecki's Theorem holds for each $f \restriction A_n$, thus the second alternative must hold for some index $k \in \omega$, that is $P \sqsubseteq f \restriction A_k$. ☐

In particular, for every Borel partition $\langle A_n \mid n \in \omega \rangle$ of $\mathbb{R}$ there is some $k \in \omega$ such that $P \restriction A_k \sqsubseteq P$ and $P \sqsubseteq P \restriction A_k$, i.e. there is some piece of the partition on which P has "maximal complexity". Proposition 6.3 is proved using AC (since Solecki's Theorem was) but, since its statement is projective, it is true also in any model of $\mathsf{ZF} + \mathsf{DC}(\mathbb{R})$ by absoluteness (see Lemma 19 of [3]). This means that, under $\mathsf{ZF} + \mathsf{DC}(\mathbb{R})$, for every Borel partition $\langle A_n \mid n \in \omega \rangle$ of $\mathbb{R}$ there is some $k \in \omega$ such that $P \restriction A_k$ is $\mathbf{\Delta}^0_\omega$ but not $\mathbf{\Delta}^0_n$ (for any $n \in \omega$), and thus Conjecture 2 fails for $\xi = \omega$.

Therefore we have proved that the Jayne-Rogers Theorem does not admit any generalization that holds for *every* level of the Borel hierarchy. Nevertheless, we have also a positive result: in fact Theorem 4.7 implies that we can extend its corollary mentioned above (assuming at least $\mathsf{SLO}^{\mathsf{D}^{\mathsf{W}}_\xi} + \neg\mathsf{FS}$) for *every* possible index $\xi \leq \omega_1$, that is we can prove that $\mathsf{D}_\xi \simeq \mathsf{D}^{\mathsf{W}}_\xi$ (clearly this is nontrivial, as we have seen, for $\xi \geq 3$). Thus, in particular, $\leq_{\mathsf{D}^{\mathsf{W}}_\xi}$ and $\leq_{\mathsf{D}_\xi}$ give rise to the same structure for every countable ξ. The same is true (under $\mathsf{SLO}^{\mathcal{F}} + \neg\mathsf{FS}$) if we replace $\mathsf{D}^{\mathsf{W}}_\xi$ with any Borel-amenable set of reductions $\mathcal{F}$ of level ξ, that is, by previous observations, for any $\mathcal{F}$ such that $\mathsf{D}^{\mathsf{Lip}}_\xi \subseteq \mathcal{F} \subseteq \mathsf{D}_\xi$ (and in this case it is easy to check that we have also $\mathrm{Sat}(\mathcal{F}) = \mathsf{D}_\xi$). To appreciate this result once again, note that for *every* $\xi \leq \omega_1$

(hence also for the simplest cases $\xi = 1, 2$) we have that $\mathsf{D}^{\mathsf{Lip}}_\xi \subsetneq \mathsf{D}^{\mathsf{W}}_\xi$ and thus also $\mathsf{D}^{\mathsf{Lip}}_\xi \subsetneq \mathsf{D}_\xi$ (therefore we get other examples of Borel-amenable sets of reductions for each level). In fact it is easy to check that $f \colon \mathbb{R} \to \mathbb{R} \colon x \mapsto \langle x(2n+1) \mid n \in \omega \rangle$ is a *uniformly* continuous function such that $f \restriction \mathbf{N}_s$ is not Lipschitz for any $s \in {}^{<\omega}\omega$. Since by the Baire Category Theorem for *any* partition $\langle A_n \mid n \in \omega \rangle$ of $\mathbb{R}$ there must be some $n_0 \in \omega$ and a sequence $s \in {}^{<\omega}\omega$ such that $\mathbf{N}_s \subsetneq \mathrm{Cl}(A_{n_0})$, using an argument similar to the one of Claim 6.2.1 one can check that if $g \colon \mathbb{R} \to \mathbb{R}$ is such that $f \restriction A_{n_0} = g \restriction A_{n_0}$ then $g \notin \mathsf{Lip}$: thus, in particular, $f \notin \mathsf{D}^{\mathsf{Lip}}_\xi$ for any $\xi \leq \omega_1$.

7. Some construction principles: nonselfdual successor degrees

We have seen that if $\mathcal{F} \in \mathsf{BAR}$ then the structure of degrees associated to it looks like the Wadge one. We now want to go further and show how to construct, given a selfdual degree, its successor degree(s) (the successor of a nonselfdual pair can easily be obtained with the $\oplus$ operation, see Theorem 3.1). Let $\xi \leq \omega_1$ be the level of $\mathcal{F}$. If $\xi = \omega_1$ we can appeal to the fact that $\mathcal{F} \simeq \mathsf{Bor}$ and that the case $\mathcal{F} = \mathsf{Bor}$ has been already treated in [4], hence we have only to consider the case $\xi < \omega_1$.

From this point on we will assume $\mathsf{SLO}^{\mathsf{L}}+\neg\mathsf{FS}+\mathsf{DC}(\mathbb{R})$ *for the rest of this Section.* Following [4] again, at the beginning it is convenient to deal with $\mathcal{F}$-pointclasses rather than $\mathcal{F}$-degrees (but we will show later in this Section how to avoid them and directly construct successor degrees). Notice that every $\mathcal{F}$-pointclass is also a boldface pointclass, since by Theorem 4.7 we have $\mathcal{F} \simeq \mathsf{D}_\xi \supseteq \mathsf{D}_1$. First note that from the analysis of the previous Sections the first nontrivial (i.e. different from $\{\emptyset\}$ and $\{\mathbb{R}\}$) $\mathcal{F}$-pointclass is $\mathbf{\Delta}^0_\xi$ (which is selfdual), and it is followed by the nonselfdual pointclasses $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$. Moreover one can easily check that $\mathbf{\Gamma}$ is a nonselfdual $\mathcal{F}$-pointclass if and only if $\mathbf{\Gamma} = \{B \subseteq \mathbb{R} \mid B \leq_{\mathsf{L}} A\}$ for some $A \subseteq \mathbb{R}$ such that $A \nleq_{\mathcal{F}} \neg A$ if and only if $\mathbf{\Gamma}$ is an $\mathcal{F}$-pointclass which admits a universal set. So let $\mathbf{\Gamma}$ be any nonselfdual $\mathcal{F}$-pointclass and let $A \nleq_{\mathcal{F}} \neg A$ be such that $\mathbf{\Gamma} = \{B \subseteq \mathbb{R} \mid B \leq_{\mathsf{L}} A\}$. Define

$$\mathbf{\Gamma}^* = \{(F \cap X) \cup (F' \setminus X') \mid F, F' \in \mathbf{\Sigma}^0_\xi, F \cap F' = \emptyset \text{ and } X, X' \in \mathbf{\Gamma}\}$$

and $\mathbf{\Delta}^* = \mathbf{\Gamma}^* \cap (\mathbf{\Gamma}^*)\breve{}$. Using the fact that $\mathbf{\Sigma}^0_\xi$ has the reduction property, we can argue as in [4] to show that $\mathbf{\Gamma}^*$ is a nonselfdual $\mathcal{F}$-pointclass which contains both $\mathbf{\Gamma}$ and $\breve{\mathbf{\Gamma}}$ and is such that $\mathbf{\Delta}^* = \{B \subseteq \mathbb{R} \mid B \leq_{\mathcal{F}} A \oplus \neg A\}$. Therefore $\{\mathbf{\Gamma}^* \setminus \mathbf{\Delta}^*, (\mathbf{\Gamma}^*)\breve{} \setminus \mathbf{\Delta}^*\}$ is the first nonselfdual pair above $[A \oplus \neg A]_{\mathcal{F}}$, i.e. it is formed by the successor degrees of $[A \oplus \neg A]_{\mathcal{F}} = \mathbf{\Delta}^* \setminus (\mathbf{\Gamma} \cup \breve{\mathbf{\Gamma}})$. Similarly, if $\langle \mathbf{\Gamma}_n \mid n \in \omega \rangle$ is a strictly increasing sequence of nonselfdual $\mathcal{F}$-pointclasses and $\mathbf{\Gamma}_n = \{B \subseteq \mathbb{R} \mid B \leq_{\mathsf{L}} A_n\}$, we can define

$$\mathbf{\Lambda} = \Big\{ \bigcup\nolimits_{n \in \omega} (F_n \cap X_n) \mid F_n \in \mathbf{\Sigma}^0_\xi, F_n \cap F_m = \emptyset \text{ if } n \neq m, \text{ and } X_n \in \mathbf{\Gamma}_n \text{ for every } n \in \omega \Big\}$$

and $\mathbf{\Delta} = \mathbf{\Lambda} \cap \breve{\mathbf{\Lambda}}$. Using the generalized reduction property of $\mathbf{\Sigma}^0_\xi$, one can prove again that $\mathbf{\Lambda}$ is a nonselfdual $\mathcal{F}$-pointclass wich contains each $\mathbf{\Gamma}_n$ and such that $\mathbf{\Delta} = \{B \subseteq \mathbb{R} \mid B \leq_{\mathcal{F}} \bigoplus_n A_n\}$. Thus $\{\mathbf{\Lambda} \setminus \mathbf{\Delta}, \breve{\mathbf{\Lambda}} \setminus \mathbf{\Delta}\}$ is the first nonselfdual pair above $[\bigoplus_n A_n]_{\mathcal{F}}$ and is formed by the successor degrees of $[\bigoplus_n A_n]_{\mathcal{F}} = \mathbf{\Delta} \setminus (\bigcup_n \mathbf{\Gamma}_n)$.

This analysis allows us to give a complete description of the first ω_1 levels of the $\leq_{\mathcal{F}}$ hierarchy: in particular, one can inductively show that the α-th pair of nonselfdual $\mathcal{F}$-pointclasses (for $\alpha < \omega_1$) is formed by α-$\mathbf{\Sigma}^0_\xi$ and its dual (for the definition of the difference pointclasses α-$\mathbf{\Gamma}$ see [7]).

Now put $\mathbf{\Pi}^0_{<\xi} = \bigcup_{\mu<\xi} \mathbf{\Pi}^0_\mu$ and let $A \subseteq \mathbb{R}$ be any set such that $A \leq_{\mathcal{F}} \neg A$: we can "summarize" the constructions above by showing that

$$\mathbf{\Gamma}^+(A) = \Big\{\bigcup_{n\in\omega}(F_n \cap A_n) \mid F_n \in \mathbf{\Pi}^0_{<\xi}, F_n \cap F_m = \emptyset \text{ for } n \neq m, \text{ and } A_n <_{\mathcal{F}} A \text{ for every } n \in \omega\Big\}$$

and its dual are the smallest nonselfdual $\mathcal{F}$-pointclasses which contain A. To see this, we must first consider two cases: if A is a successor with respect to $\leq_{\mathcal{F}}$ (i.e. $A \equiv_{\mathcal{F}} C \oplus \neg C$ for some $C \not\leq_{\mathcal{F}} \neg C$) then $\mathbf{\Gamma}^* \subseteq \mathbf{\Gamma}^+(A)$ (where $\mathbf{\Gamma}^*$ is obtained from $\mathbf{\Gamma} = \{B \subseteq \mathbb{R} \mid B \leq_{\mathsf{L}} C\}$ as before), while if A is limit then there is a strictly increasing sequence of nonselfdual $\mathcal{F}$-pointclasses $\mathbf{\Gamma}_n = \{B \subseteq \mathbb{R} \mid B \leq_{\mathsf{L}} A_n\}$ such that $A \equiv_{\mathcal{F}} \bigoplus_n A_n$, and it is not hard to see that $\mathbf{\Lambda} \subseteq \mathbf{\Gamma}^+(A)$, where $\mathbf{\Lambda}$ is constructed from the $\mathbf{\Gamma}_n$'s as above. Since $\mathbf{\Gamma}^+(A)$ is clearly an $\mathcal{F}$-pointclass, the result will follow if we can prove that if $B \subseteq \mathbb{R}$ is such that $B, \neg B \in \mathbf{\Gamma}^+(A)$ then $B \leq_{\mathcal{F}} A$. So let $B = \bigcup_n (F_n \cap A_n)$ and $\neg B = \bigcup_n (F'_n \cap A'_n)$. Since $\bigcup_n (F_n \cup F'_n) = \mathbb{R}$ and $\mathbf{\Sigma}^0_\xi$ has the generalized reduction property, we can find $\hat{F}_n, \hat{F}'_n \in \mathbf{\Delta}^0_\xi$ such that they form a partition of $\mathbb{R}$ and $\hat{F}_n \subseteq F_n$, $\hat{F}'_n \subseteq F'_n$ for each n. Hence

$$x \in \hat{F}_n \Rightarrow (x \in B \iff x \in A_n)$$
$$x \in \hat{F}'_n \Rightarrow (x \in B \iff x \notin A'_n),$$

and since $A_n, \neg A'_n <_{\mathcal{F}} A$ by $\mathsf{SLO}^{\mathcal{F}}$, we get $B \leq_{\mathcal{F}} A$ by Borel-amenability of $\mathcal{F}$ (see Proposition 4.5).

Now we will show how to construct the successor of an $\mathcal{F}$-selfdual degree $[A]_{\mathcal{F}}$ in a "direct" way, i.e. without considering the associated $\mathcal{F}$-pointclasses. First fix an increasing sequence of ordinals $\langle \mu_n \mid n \in \omega\rangle$ cofinal in ξ (clearly if $\xi = \nu + 1$ we can take $\mu_n = \nu$ for every $n \in \omega$), and a sequence of sets P_n such that $P_n \in \mathbf{\Pi}^0_{\mu_n} \setminus \mathbf{\Sigma}^0_{\mu_n}$. Let $\langle\cdot,\cdot\rangle \colon \omega \times \omega \to \omega$ be any bijection, e.g. $\langle n, m\rangle = 2^n(2m+1) - 1$. Then we can define the homeomorphism

$$\bigotimes \colon {}^\omega\mathbb{R} \to \mathbb{R} \colon \langle x_n \mid n \in \omega\rangle \mapsto x = \bigotimes_n x_n,$$

where $x(\langle n, m\rangle) = x_n(m)$, and, conversely, the "projections" $\pi_n \colon \mathbb{R} \to \mathbb{R}$ defined by $\pi_n(x) = \langle x(\langle n, m\rangle) \mid m \in \omega\rangle$ (clearly, every "projection" is surjective, continuous and open). Moreover, given a sequence of functions $f_n \colon \mathbb{R} \to \mathbb{R}$, we can use the homeomorphism $\bigotimes$ to define the function

$$\bigotimes(\langle f_n \mid n \in \omega\rangle) = \bigotimes_n f_n \colon \mathbb{R} \to \mathbb{R} \colon x \mapsto \bigotimes_n (f_n(x)).$$

It is not hard to check that $\bigotimes_n f_n$ is continuous if and only if all the f_n's are continuous. Now consider the set

$$\Sigma^\xi(A) = \{x \in \mathbb{R} \mid \exists n(\pi_{2n}(x) \in P_n \wedge \forall i < n(\pi_{2i}(x) \notin P_i) \wedge \pi_{2n+1}(x) \in A)\}.$$

We will prove that $\Sigma^\xi(A)$ is $\mathbf{\Gamma}^+(A)$-complete, from which it follows that $[\Sigma^\xi(A)]_{\mathcal{F}}$ is a (nonselfdual) successor of $[A]_{\mathcal{F}}$. Inductively define $F_0 = \{x \in \mathbb{R} \mid \pi_0(x) \in P_0\}$ and $F_{n+1} = \{x \in \mathbb{R} \mid \pi_{2(n+1)}(x) \in P_{n+1} \wedge \forall i \leq n(\pi_{2i}(x) \notin P_i)\}$. Clearly $F_n \in \mathbf{\Delta}^0_\xi$, $F_n \cap F_m = \emptyset$ for $n \neq m$, and $\Sigma^\xi(A) \subseteq \bigcup_n F_n$. Moreover put $A_n = \pi^{-1}_{2n+1}(A)$, and let $\langle D_{n,k} \mid k \in \omega\rangle$ be a $\mathbf{\Pi}^0_{<\xi}$-partitions of $\mathbb{R}$ such that $A_n \cap D_{n,k} <_{\mathcal{F}} A$ for every $k \in \omega$ (this partitions must exist by Theorem 5.3 if $A_n \equiv_{\mathcal{F}} A \equiv_{\mathcal{F}} \neg A$: if instead $A_n <_{\mathcal{F}} A$ simply take $D_{n,0} = \mathbb{R}$ and $D_{n,k+1} = \emptyset$). Finally, let $G_{n,m} \in \mathbf{\Pi}^0_{<\xi}$ be such that $G_{n,m} \cap G_{n,m'} = \emptyset$ if $m \neq m'$ and $F_n = \bigcup_m G_{n,m}$. Thus

$$\Sigma^\xi(A) = \bigcup_{n,m,k\in\omega}((G_{n,m} \cap D_{n,k}) \cap (D_{n,k} \cap A)),$$

and hence $\Sigma^\xi(A) \in \mathbf{\Gamma}^+(A)$ by definition.

Conversely let $B = \bigcup_k (F_k \cap A_k)$ be a generic set in $\mathbf{\Gamma}^+(A)$ and let n_k be an increasing sequence of natural numbers such that $F_k \in \mathbf{\Pi}^0_{\mu_{n_k}}$ (such a sequence must exist since the sequence μ_n is cofinal in ξ). Fix $y_i \notin P_i$ for every $i \in \omega$ and define

$$f_i = \begin{cases} \text{a continuous reduction of } F_k \text{ to } P_{n_k} & \text{if } i = n_k \\ \text{the constant function with value } y_i & \text{otherwise} \end{cases}$$

$$g_i = \begin{cases} \text{a continuous reduction of } A_n \text{ to } A & \text{if } i = n_k \\ \text{id} & \text{otherwise} \end{cases}$$

(A_n is continuously reducible to A by $\mathsf{SLO}^{\mathsf{L}}$ and the fact that $A_n <_{\mathcal{F}} A$). Finally put $h_{2i} = f_i$ and $h_{2i+1} = g_i$ for every $i \in \omega$: it is easy to check that $f = \bigotimes_i h_i$ is continuous and reduces B to $\Sigma^\xi(A)$.

In a similar way one can prove that the set

$$\Pi^\xi(A) = \Sigma^\xi(A) \cup P_\xi,$$

where $P_\xi = \{x \in \mathbb{R} \mid \forall n(\pi_{2n}(x) \notin P_n)\}$, is complete for the dual pointclass of $\mathbf{\Gamma}^+(A)$, from which it follows that $\Pi^\xi(A) \equiv_{\mathcal{F}} \neg\Sigma^\xi(A)$. This construction suggests also how to define certain games G^{W}_ξ which represent a full generalization for all the levels of the backtrack game (in the sense that the legal strategies for player **II** in G^{W}_ξ induce exactly the functions in $\mathsf{D}^{\mathsf{W}}_\xi$). This seems to be useful since it allows to use "combinatorial" arguments to prove results about the $\mathsf{D}^{\mathsf{W}}_\xi$-hierarchies (and hence, by Proposition 4.3 and Theorem 4.7, also about the degree-structure induced by *any* Borel-amenable set of reductions).

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